

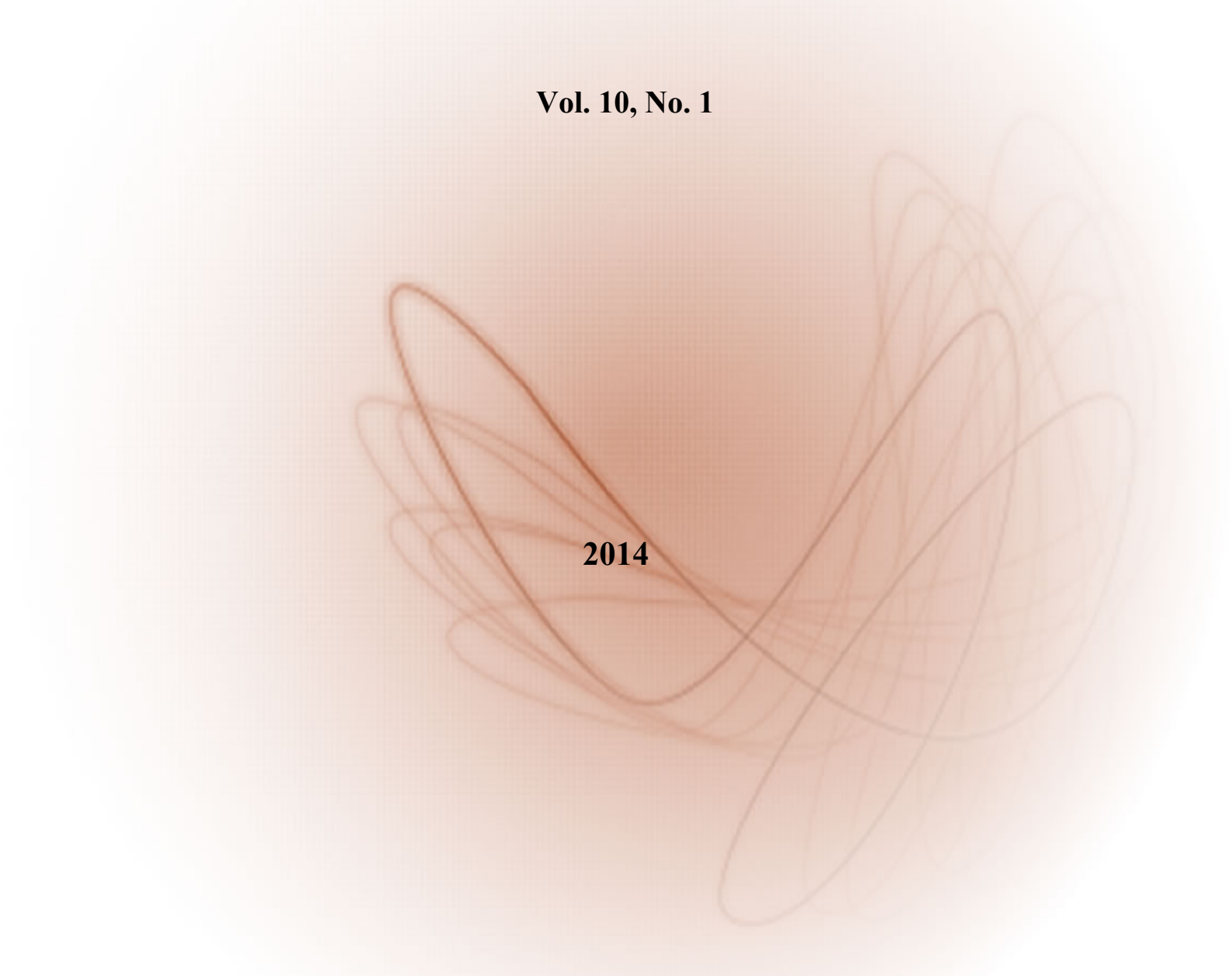


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# $\Phi$ -variational stability for discontinuous system

Xuefeng Liang<sup>†</sup> and Wansheng He<sup>‡</sup>

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**Abstract** Bounded  $\Phi$ -variation functions are development and generalization of bounded variation functions in the usual sense. The concept of Henstock-Kurzweil integral is an effective tool in dealing with highly infinite oscillation functions. In this paper, the concept of  $\Phi$ -variational stability is defined by using  $\Phi$ -function theory, the  $\Phi$ -variational stability of the bounded  $\Phi$ -variation solution to discontinuous system is discussed, and the Ljapunov type theorems for  $\Phi$ -variational stability and asymptotically  $\Phi$ -variational stability of the bounded  $\Phi$ -variation solution are established. These results are essential generalization of variation stability of bounded variation solution of this system, and the certain foundation is laid in the research of highly infinite oscillation functions.

**Keywords** Discontinuous system, Henstock-Kurzweil integral, bounded  $\Phi$ -variation solution,  $\Phi$ -variational stability, Ljapunov function.

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## §1. Introduction

Consider the general discontinuous system

$$x' = f(t, x), \quad (*)$$

where  $x = (x_1, x_2, \dots, x_n)^T$ ,  $x' = \frac{dx}{dt}$ ,  $f : G \rightarrow R^n$ ,  $G$  is an open region in  $R^{n+1}$ .  $(*)$  is called discontinuous system if  $f$  is a function with some discontinuity. In [1] and [2], the existence, uniqueness and stability for the solution of Caratheodory system and Filippov system were obtained by using the Lebesgue integral and the solutions obtained are absolute continuous functions. The Henstock-Kurzweil integral was introduced by Henstock and Kurzweil independently during 1957-1958, the existence, uniqueness and variational stability for the bounded variation solutions of a class of discontinuous system were discussed by using the Henstock-Kurzweil integral in [3, 4].

The functions of bounded  $\Phi$ -variation were introduced by Musielak and Orlicz [5,6], which are generalization and development of functions of bounded variation in usual sense. The

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existence theorem of bounded  $\Phi$ -variation solutions for a class of discontinuous system is established in [7]. In this paper, the concept of  $\Phi$ -variational stability of bounded  $\Phi$ -variation solutions for discontinuous system is given and the stability is discussed through making use of Ljapunov function. The Ljapunov type theorems for  $\Phi$ -variational stability and asymptotically  $\Phi$ -variational stability of the bounded  $\Phi$ -variation solutions are established. These results are essential generalization of variation stability of bounded variation solution to discontinuous system in [4].

## §2. Preliminaries and definitions

**Definition 2.1.**<sup>[3,4,7,8]</sup> A function  $x : [a, b] \rightarrow R^n$  is called Henstock-Kurzweil integrable over  $[a, b]$ , if there is a  $A \in R^n$  such that given  $\varepsilon > 0$ , there is a positive function  $\delta : [a, b] \rightarrow (0, +\infty)$  such that for any  $\delta(t)$ -fine partition  $D : a = t_0 < t_1 < \cdots < t_k = b$  and  $\{\xi_1, \xi_2, \dots, \xi_k\}$  satisfying  $\xi_j \in [t_{j-1}, t_j] \subset [\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)]$ , we have

$$\left\| \sum_{j=1}^k x(\xi_j)(t_j - t_{j-1}) - A \right\| < \varepsilon.$$

$A$  is called the Henstock-Kurzweil integral of  $x(t)$  over  $[a, b]$  and will be denoted by  $\int_a^b x(t)dt$ .

**Definition 2.2.**<sup>[3,4,7,8]</sup> A function  $x : [a, b] \rightarrow R^n$  is called H-K-Stieltjes integral over  $[a, b]$ , function  $y : [a, b] \rightarrow R^n$ , if there is a  $A \in R^n$  such that given  $\varepsilon > 0$ , there is a positive function  $\delta : [a, b] \rightarrow (0, +\infty)$  such that for any  $\delta(t)$ -fine partition  $D : a = t_0 < t_1 < \cdots < t_k = b$  and  $\{\xi_1, \xi_2, \dots, \xi_k\}$  satisfying  $\xi_j \in [t_{j-1}, t_j] \subset [\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)]$ , we have

$$\left\| \sum_{j=1}^k x(\xi_j)(y(t_j) - y(t_{j-1})) - A \right\| < \varepsilon.$$

$A$  is called the H-K-Stieltjes integral of  $x(t)$  over  $[a, b]$  and will be denoted by  $\int_a^b x(t)dy(t)$ .

Let  $\Phi(u)$  denote a continuous and increasing function defined for  $u \geq 0$  with  $\Phi(0) = 0$ ,  $\Phi(u) > 0$  for  $u > 0$ , and satisfying the following conditions:

- ( $\Delta_1$ ) There exist  $u_0 \geq 0$  and  $a > 0$  such that  $\Phi(2u) \leq a\Phi(u)$  for  $0 < u \leq u_0$ ;
- ( $\Delta_2$ )  $\Phi(u)$  is a convex function.

Let  $[a, b] \subset R$ ,  $-\infty < a < b < +\infty$ . We consider the function  $x : [a, b] \rightarrow R^n$ ,  $x(t)$  is of bounded  $\Phi$ -variation over  $[a, b]$  if for any partition  $\pi : a = t_0 < t_1 < \cdots < t_m = b$ , we have  $V_\Phi(x; [a, b]) = \sup_\pi \sum_{i=1}^m \Phi(\|x(t_i) - x(t_{i-1})\|) < +\infty$ ,  $V_\Phi(x; [a, b])$  is called  $\Phi$ -variation of  $x(t)$  over  $[a, b]$ . We always assume  $\Phi(u)$  satisfying ( $\Delta_1$ ) and ( $\Delta_2$ ).

Given  $c > 0$ , we denote  $B_c = \{x \in R^n; \|x\| < c\}$ . Let  $I \subset (a, b) \subset R$  be an interval with  $-\infty < a < b < +\infty$ , and set  $G = B_c \times I$ .

**Definition 2.3.**<sup>[3,4,7]</sup> A Caratheodory function  $f : G \rightarrow R^n$  belongs to the class  $\mathcal{V}_\Phi(G, h, \omega)$ , if

- (1) There exists a positive function  $\delta : [a, b] \rightarrow (0, +\infty)$ , satisfying  $\tau \in [u, v] \subset [\tau - \delta(\tau), \tau + \delta(\tau)] \subset I$  and  $x \in \bar{B}$  for every  $[u, v]$ , we have

$$\|f(x, \tau)(v - u)\| \leq \Phi(|h(v) - h(u)|).$$

(2) Satisfying  $\tau \in [u, v] \subset [\tau - \delta(\tau), \tau + \delta(\tau)] \subset I$  and  $x, y \in \bar{B}$  for every  $[u, v]$ , we have

$$\|f(x, \tau) - f(y, \tau)\| (v - u) \leq \omega(\|x - y\|) \Phi(\|h(v) - h(u)\|),$$

where  $h : [a, b] \rightarrow R$  is an increasing function and continuous from the left on the interval  $[a, b]$ ,  $\omega : [0, +\infty) \rightarrow R$  is a continuous and increasing function with  $\omega(0) = 0$ .

(3) The function  $f(\psi(t), t)$  is  $H - K$  integrable on  $[\alpha, \beta]$  for every step function  $\psi(t)$  on  $[\alpha, \beta] \subset [a, b]$ .

**Lemma 2.1.**<sup>[3,4,7]</sup> Assume that  $f : G \rightarrow R^n$  satisfies the condition (1) of Definition 2.3, if  $x : [\alpha, \beta] \rightarrow R^n$ ,  $[\alpha, \beta] \subset (a, b)$  is such that  $(x(t), t) \in G$  for every  $t \in [\alpha, \beta]$  and if the Kurzweil integral  $\int_{\alpha}^{\beta} f(x(t), t) dt$  exists, then for every pair  $s_1, s_2 \in [\alpha, \beta]$  the inequality  $\|\int_{\alpha}^{\beta} f(x(t), t) dt\| \leq V_{\Phi}(h; [s_1, s_2])$  holds.

**Corollary 2.1.**<sup>[3,4,7]</sup> Assume that  $f : G \rightarrow R^n$  satisfies the condition (1) of Definition 2.3, if  $x : [\alpha, \beta] \rightarrow R^n$ ,  $[\alpha, \beta] \subset (a, b)$  is a solution of (\*), then  $x$  is of bounded  $\Phi$ -variation and  $V_{\Phi}(x; [\alpha, \beta]) \leq \Phi(V_{\Phi}(h; [\alpha, \beta])) < +\infty$ . Moreover every point in  $[\alpha, \beta]$  at which the function  $h$  is continuous from the left is a left continuity point of the solution  $x : [\alpha, \beta] \rightarrow R^n$ .

**Lemma 2.2.**<sup>[4,8]</sup> Assume that  $-\infty < a < b < +\infty$  and that  $f, g : [a, b] \rightarrow R$  are functions which are continuous from the left in  $[a, b]$ . If for every  $\sigma \in [a, b]$  there exists  $\delta(\sigma) > 0$  such that for every  $\eta \in (0, \delta(\sigma))$  the inequality  $f(\sigma + \eta) - f(\sigma) \leq g(\sigma + \eta) - g(\sigma)$  holds, then  $f(s) - f(a) \leq g(s) - g(a)$  for all  $s \in [a, b]$ .

If  $f(0, t) = 0$  for every  $t \in [0, +\infty)$  then the function  $x$  given by  $x(t) = 0$  for  $t \geq 0$  is a solution of (\*) on the whole half-axis  $[0, +\infty)$ .

**Definition 2.4.** The solution  $x \equiv 0$  of (\*) is called  $\Phi$ -variationally stable if for every  $\varepsilon > 0$  there exists  $\delta = \delta(\varepsilon) > 0$  such that if  $y : [t_0, t_1] \rightarrow B_c$ ,  $0 \leq t_0 < t_1 < +\infty$  is a function of bounded  $\Phi$ -variation on  $[t_0, t_1]$ , continuous from the left on  $(t_0, t_1]$  with  $\Phi(\|y(t_0)\|) < \delta$  and  $V_{\Phi}((y(s) - \int_{t_0}^s f(y(t), t) dt); [t_0, t_1]) < \delta$ , then we have  $\Phi(\|y(t)\|) < \varepsilon$ .

**Definition 2.5.** The solution  $x \equiv 0$  of (\*) is called  $\Phi$ -variationally attracting if there exists  $\delta_0 > 0$  and for every  $\varepsilon > 0$  there is a  $T = T(\varepsilon) \geq 0$  and  $\gamma = \gamma(\varepsilon) > 0$  such that if  $y : [t_0, t_1] \rightarrow B_c$ ,  $0 \leq t_0 < t_1 < +\infty$  is a function of bounded  $\Phi$ -variation on  $[t_0, t_1]$ , continuous from the left on  $(t_0, t_1]$  with  $\Phi(\|y(t_0)\|) < \delta_0$  and  $V_{\Phi}((y(s) - \int_{t_0}^s f(y(t), t) dt); [t_0, t_1]) < \gamma$ , then  $\Phi(\|y(t)\|) < \varepsilon$  for all  $t \in [t_0, t_1] \cap [t_0 + T(\varepsilon), +\infty)$  and  $t_0 \geq 0$ .

**Definition 2.6.** The solution  $x \equiv 0$  of (\*) is called  $\Phi$ -variationally asymptotically stable if it is  $\Phi$ -variationally stable and  $\Phi$ -variationally attracting.

### §3. Main results

**Theorem 3.1.** Assume that  $V : [0, +\infty) \times R^n \rightarrow R$  is such that for every  $x \in R^n$  the function  $V(\cdot, x) : [0, +\infty) \rightarrow R$  is continuous from the left in  $(0, +\infty)$  and

(1) The inequality

$$|V(t, x) - V(t, y)| \leq L\Phi(\|x - y\|) \tag{1}$$

holds for  $x, y \in R^n$ ,  $t \in [0, +\infty)$  with a constant  $L > 0$ ;

(2) There exists a real function  $H : R^n \rightarrow R$  such that for every solution  $x : (\alpha, \beta) \rightarrow R^n$  of the discontinuous system  $(*)$  on  $(\alpha, \beta) \subset [0, +\infty)$ , we have

$$\lim_{\eta \rightarrow 0+} \sup \frac{V(t + \eta, x(t + \eta)) - V(t, x(t))}{\eta} \leq H(x(t)) \quad (2)$$

for  $t \in (\alpha, \beta)$ .

If  $y : [t_0, t_1] \rightarrow R^n$ ,  $0 \leq t_0 < t_1 < +\infty$  is continuous from the left on  $(t_0, t_1]$  and of bounded  $\Phi$ -variation on  $[t_0, t_1]$ , then the inequality

$$V(t_1, y(t_1)) \leq V(t_0, y(t_0)) + NV_{\Phi}((y(s) - \int_{t_0}^s f(y(t), t)dt); [t_0, t_1]) + R(t_1 - t_0) \quad (3)$$

holds, where  $R = \sup_{t \in [t_0, t_1]} H(x(t))$ ,  $N = L\chi(a')$ ,  $(\chi(a'))$  was defined by Theorem 1.2 in [5].

**Proof.** Let  $y : [t_0, t_1] \rightarrow R^n$ ,  $0 \leq t_0 < t_1 < +\infty$  be a function of bounded  $\Phi$ -variation on  $[t_0, t_1]$  and continuous from the left on  $(t_0, t_1]$  and let  $\sigma \in [t_0, t_1]$  be an arbitrary point. It is clear that the function  $V(t, y(t)) : [t_0, t_1] \times R^n \rightarrow R$  is continuous from the left on  $(t_0, t_1]$ .

Assume that  $x : [\sigma, \sigma + \eta_1(\sigma)] \rightarrow R^n$  is a solution of bounded  $\Phi$ -variation of  $(*)$  on the interval  $[\sigma, \sigma + \eta_1(\sigma)]$ ,  $\eta_1 > 0$  with the initial condition  $x(\sigma) = y(\sigma)$ . The existence of such a solution is guaranteed by Theorem 3.3 in [7]. By (1), we have

$$\begin{aligned} V(\sigma + \eta, y(\sigma + \eta)) - V(\sigma + \eta, x(\sigma + \eta)) &\leq L\Phi(\|y(\sigma + \eta) - x(\sigma + \eta)\|) \\ &= L\Phi(\|y(\sigma + \eta) - y(\sigma) - \int_{\sigma}^{\sigma + \eta} f(x(t), t)dt\|) \end{aligned}$$

for every  $\eta \in [0, \eta_1(\sigma)]$ .

By (2), we obtain

$$\begin{aligned} &V(\sigma + \eta, y(\sigma + \eta)) - V(\sigma, x(\sigma)) \\ &= V(\sigma + \eta, y(\sigma + \eta)) - V(\sigma + \eta, x(\sigma + \eta)) + V(\sigma + \eta, x(\sigma + \eta)) - V(\sigma, x(\sigma)) \\ &\leq L\Phi(\|y(\sigma + \eta) - y(\sigma) - \int_{\sigma}^{\sigma + \eta} f(x(t), t)dt\|) + \eta H(x(\sigma)) \\ &\leq L\Phi(\|y(\sigma + \eta) - y(\sigma) - \int_{\sigma}^{\sigma + \eta} f(x(t), t)dt\|) + \eta R + \eta \varepsilon, \end{aligned}$$

where  $\varepsilon > 0$  is arbitrary and  $\eta \in (0, \eta_2(\sigma))$  with  $\eta_2(\sigma) \leq \eta_1(\sigma)$ ,  $\eta_2(\sigma) > 0$  is sufficiently small.

Denote

$$\Psi(s) = y(s) - \int_{t_0}^s f(y(t), t)dt, s \in [t_0, t_1],$$

the function  $\Psi : [t_0, t_1] \rightarrow R^n$  is of bounded  $\Phi$ -variation and continuous from the left on  $(t_0, t_1]$ .

By Theorem 1.2, Theorem 1.11 and Theorem 1.17 in [5], we have

$$\begin{aligned} &V(\sigma + \eta, y(\sigma + \eta)) - V(\sigma, x(\sigma)) \\ &\leq L\Phi(\|y(\sigma + \eta) - y(\sigma) - \int_{\sigma}^{\sigma + \eta} f(y(t), t)dt\| + \|\int_{\sigma}^{\sigma + \eta} [f(y(t), t) - f(x(t), t)]dt\|) \\ &\quad + \eta R + \eta \varepsilon \leq L\chi(a')\{\Phi(\|y(\sigma + \eta) - y(\sigma) - \int_{\sigma}^{\sigma + \eta} f(y(t), t)dt\|) \\ &\quad + \Phi(\|\int_{\sigma}^{\sigma + \eta} [f(y(t), t) - f(x(t), t)]dt\|)\} + \eta R + \eta \varepsilon \end{aligned}$$

$$\begin{aligned}
&\leq L\chi(a')V_{\Phi}((y(s) - \int_{\sigma}^s f(y(t), t)dt); [\sigma, \sigma + \eta]) + \eta R + \eta\varepsilon \\
&\quad + L\chi(a')\Phi(\|\int_{\sigma}^{\sigma+\eta}[f(y(t), t) - f(x(t), t)]dt\|) \\
&\leq NV_{\Phi}(\Psi; [\sigma, \sigma + \eta]) + \eta R + \eta\varepsilon + N\Phi(\|\int_{\sigma}^{\sigma+\eta}[f(y(t), t) - f(x(t), t)]dt\|) \\
&\leq N(V_{\Phi}(\Psi; [\sigma, \sigma + \eta]) - V_{\Phi}(\Psi; [\sigma, \sigma])) + \eta R + \eta\varepsilon \\
&\quad + N\Phi(\|\int_{\sigma}^{\sigma+\eta}[f(y(t), t) - f(x(t), t)]dt\|)
\end{aligned} \tag{4}$$

for every  $\eta \in (0, \eta_2(\sigma))$ .

Since  $f \in \mathcal{V}_{\Phi}(G, h, \omega)$  and by Definition 2.1, there exists a positive function  $\delta(\tau) : [\sigma, \sigma + \eta] \rightarrow (0, +\infty)$  for  $\varepsilon > 0$  such that any  $\delta(\tau)$ -fine partition  $D = \{(\tau_j; [\alpha_{j-1}, \alpha_j]), j = 1, 2, \dots, m\}$  on  $[\sigma, \sigma + \eta]$ , we have

$$\begin{aligned}
&\|\int_{\sigma}^{\sigma+\eta}[f(y(t), t) - f(x(t), t)]dt\| \\
&\leq \|\int_{\sigma}^{\sigma+\eta}[(y(t), t) - f(x(t), t)]dt - \sum_{j=1}^m[f(y(\tau_j), \alpha_j) - f(x(\tau_j), \alpha_j) \\
&\quad - f(y(\tau_j), \alpha_{j-1}) + f(x(\tau_j), \alpha_{j-1})]\| \\
&\quad + \|\sum_{j=1}^m[f(y(\tau_j), \alpha_j) - f(x(\tau_j), \alpha_j) - f(y(\tau_j), \alpha_{j-1}) + f(x(\tau_j), \alpha_{j-1})]\| \\
&< \frac{\varepsilon}{2} + \sum_{j=1}^m \omega(\|y(\tau_j) - x(\tau_j)\|)\Phi(\|h(\alpha_j) - h(\alpha_{j-1})\|).
\end{aligned} \tag{5}$$

By Theorem 1.3 and Theorem 1.17 in [5], we have

$$\begin{aligned}
&\sum_{j=1}^m \omega(\|y(\tau_j) - x(\tau_j)\|)\Phi(\|h(\alpha_j) - h(\alpha_{j-1})\|) \\
&= \sum_{j=1}^m \omega(\|y(\tau_j) - x(\tau_j)\|)V_{\Phi}(h; [\alpha_{j-1}, \alpha_j]) \\
&\leq \sum_{j=1}^m \omega(\|y(\tau_j) - x(\tau_j)\|)[V_{\Phi}(h; [\sigma, \alpha_j]) - V_{\Phi}(h; [\sigma, \alpha_{j-1}])].
\end{aligned}$$

Denote  $U(t) = V_{\Phi}(h; [\sigma, t])$  for  $\sigma \leq t \leq \sigma + \eta$ . By definitions of  $\Phi(u)$  and  $h$ ,  $U(t)$  is a function of non-navigate and nondecreasing on  $[\sigma, \sigma + \eta]$  and continuous from the left on  $(\sigma, \sigma + \eta]$ , then by (5) we have

$$\begin{aligned}
&\sum_{j=1}^m \omega(\|y(\tau_j) - x(\tau_j)\|)\Phi(\|h(\alpha_j) - h(\alpha_{j-1})\|) \\
&\leq \sum_{j=1}^m \omega(\|y(\tau_j) - x(\tau_j)\|)(U(\alpha_j) - U(\alpha_{j-1})) \\
&\leq \sum_{j=1}^m \omega(\|y(\tau_j) - x(\tau_j)\|)(U(\alpha_j) - U(\alpha_{j-1})) - \int_{\sigma}^{\sigma+\eta} \omega(\|y(t) - x(t)\|)dU(t) \\
&\quad + \int_{\sigma}^{\sigma+\eta} \omega(\|y(t) - x(t)\|)dU(t) < \frac{\varepsilon}{2} + \int_{\sigma}^{\sigma+\eta} \omega(\|y(t) - x(t)\|)dU(t).
\end{aligned} \tag{6}$$

By (5), (6), Theorem 1.16 in [8] and arbitrary  $\varepsilon > 0$ , we have

$$\begin{aligned}
&\|\int_{\sigma}^{\sigma+\eta}[f(y(t), t) - f(x(t), t)]dt\| \leq \int_{\sigma}^{\sigma+\eta} \omega(\|y(t) - x(t)\|)dU(t) \\
&= \lim_{\alpha \rightarrow 0+} [\int_{\sigma}^{\sigma+\alpha} \omega(\|y(t) - x(t)\|)dU(t) + \int_{\sigma+\alpha}^{\sigma+\eta} \omega(\|y(t) - x(t)\|)dU(t)] \\
&= \omega(\|y(\sigma) - x(\sigma)\|)(U(\sigma+) - U(\sigma)) + \lim_{\alpha \rightarrow 0+} \int_{\sigma+\alpha}^{\sigma+\eta} \omega(\|y(t) - x(t)\|)dU(t)
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\alpha \rightarrow 0+} \int_{\sigma+\alpha}^{\sigma+\eta} \omega(\|y(t) - x(t)\|) dU(t) \\
&\leq \sup_{\rho \in [\sigma, \sigma+\eta]} \omega(\|y(\rho) - x(\rho)\|) \lim_{\alpha \rightarrow 0+} (U(\sigma+\eta) - U(\sigma+\alpha)) \\
&= \sup_{\rho \in [\sigma, \sigma+\eta]} \omega(\|y(\rho) - x(\rho)\|) (U(\sigma+\eta) - U(\sigma+)). \tag{7}
\end{aligned}$$

Let  $\rho \in [\sigma, \sigma + \eta_2(\sigma)]$ , we have  $y(\rho) - x(\rho) = y(\rho) - y(\sigma) - \int_{\sigma}^{\rho} f(x(t), t) dt$ , and therefore

$$\begin{aligned}
\lim_{\rho \rightarrow \sigma+} (y(\rho) - x(\rho)) &= y(\sigma+) - y(\sigma) - \lim_{\rho \rightarrow \sigma+} (f(x(\sigma), \rho) - f(x(\sigma), \sigma)) \\
&= y(\sigma+) - y(\sigma) - (f(x(\sigma), \sigma+) - f(x(\sigma), \sigma)) = \Psi(\sigma+) - \Psi(\sigma),
\end{aligned}$$

and also

$$\lim_{\rho \rightarrow \sigma+} \|y(\rho) - x(\rho)\| = \|\Psi(\sigma+) - \Psi(\sigma)\|. \tag{8}$$

For every  $\varepsilon > 0$  denote

$$\beta = \frac{\varepsilon}{N(U(t_1) - U(t_0) + 1)} > 0 \tag{9}$$

and assume that  $r = r(\beta) > 0$  is such that  $\omega(r) < \beta$ . Further, we choice  $\alpha \in (0, \frac{r}{2})$ .

Since (8) holds, there is an  $\eta_3(\sigma) \in (0, \eta_2(\sigma))$  such that

$$\|y(\rho) - x(\rho)\| \leq \|\Psi(\sigma+) - \Psi(\sigma)\| + \alpha,$$

and also

$$\omega(\|y(\rho) - x(\rho)\|) \leq \omega(\|\Psi(\sigma+) - \Psi(\sigma)\| + \alpha) \tag{10}$$

for  $\rho \in (\sigma, \sigma + \eta_3(\sigma))$ .

Denote  $P(\beta) = \{\sigma \in [t_0, t_1]; \|\Psi(\sigma+) - \Psi(\sigma)\| \geq \frac{r}{2}\}$ , since  $\Psi$  is of bounded  $\Phi$ -variation on  $[t_0, t_1]$ , the set  $P(\beta)$  is finite and we denote by  $n(\beta)$  the number of elements of  $P(\beta)$ .

If  $\sigma \in [t_0, t_1] \setminus P(\beta)$  and  $\rho \in (\sigma, \sigma + \eta_3(\sigma))$  then by (10) we have

$$\omega(\|y(\rho) - x(\rho)\|) \leq \omega(\|\Psi(\sigma+) - \Psi(\sigma)\| + \alpha) \leq \omega(\frac{r}{2} + \alpha) < \omega(\frac{r}{2} + \frac{r}{2}) = \omega(r) < \beta,$$

and by (7) also

$$\left\| \int_{\sigma}^{\sigma+\eta} [f(y(t), t) - f(x(t), t)] dt \right\| \leq \beta (U(\sigma+\eta) - U(\sigma+)) \tag{11}$$

whenever  $\eta \in (0, \eta_3(\sigma))$ .

If  $\sigma \in [t_0, t_1] \cap P(\beta)$  then there exists  $\eta_4(\sigma) \in (0, \eta_3(\sigma))$  such that for  $\eta \in (0, \eta_4(\sigma))$  we have

$$U(\sigma+\eta) - U(\sigma+) = |U(\sigma+\eta) - U(\sigma+)| < \frac{\beta}{(n(\beta) + 1)\omega(\|\Psi(\sigma+) - \Psi(\sigma)\| + \alpha)}$$

and  $(\sigma, \sigma + \eta_4(\sigma)) \cap P(\beta) = \emptyset$ . Hence (7) and (10) yield

$$\left\| \int_{\sigma}^{\sigma+\eta} [f(y(t), t) - f(x(t), t)] dt \right\|$$

$$\leq \omega(\|\Psi(\sigma+) - \Psi(\sigma)\| + \alpha) \frac{\beta}{(n(\beta) + 1)\omega(\|\Psi(\sigma+) - \Psi(\sigma)\| + \alpha)} = \frac{\beta}{n(\beta) + 1} \quad (12)$$

for every  $\eta \in (\sigma, \sigma + \eta_4(\sigma))$ .

Denote  $\tilde{U}_\alpha(t) = \frac{\beta}{n(\beta)+1} \sum_{\sigma \in P(\beta)} I_\sigma(t)$  for  $t \in [t_0, t_1]$  where  $I_\sigma(t) = 0$  for  $t \leq \sigma$  and  $I_\sigma(t) = 1$  for  $t > \sigma$ . The function  $\tilde{U}_\alpha : [t_0, t_1] \rightarrow R$  is nondecreasing and continuous from the left and

$$\tilde{U}_\alpha(t_1) - \tilde{U}_\alpha(t_0) = \frac{\beta}{n(\beta) + 1} n(\beta) < \beta. \quad (13)$$

The points of discontinuity of the function  $\tilde{U}_\alpha$  are clearly only the points belonging to  $P(\beta)$ , and for  $t \in P(\beta)$  we have

$$\tilde{U}_\alpha(t+) - \tilde{U}_\alpha(t) = \frac{\beta}{n(\beta) + 1}, \quad t \in [t_0, t_1].$$

Using the function  $\tilde{U}_\alpha$ , we can set  $U_\alpha(t) = \beta U_c(t) + \tilde{U}_\alpha(t)$  for  $t \in [t_0, t_1]$  where by  $U_c$  denote the continuous part of the function  $U$ , then the function  $U_\alpha$  is nondecreasing and continuous from the left on  $[t_0, t_1]$  and by (9) and (13) we obtain

$$\begin{aligned} U_\alpha(t_1) - U_\alpha(t_0) &= \beta(U_c(t_1) - U_c(t_0)) + \tilde{U}_\alpha(t_1) - \tilde{U}_\alpha(t_0) \\ &< \beta[U(t_1) - U(t_0) + 1] = \frac{\varepsilon}{N}. \end{aligned} \quad (14)$$

If  $\sigma \in [t_0, t_1] \setminus P(\beta)$  then set  $\delta(\sigma) = \eta_3(\sigma) > 0$  and if  $\sigma \in [t_0, t_1] \cap P(\beta)$  then set  $\delta(\sigma) = \eta_4(\sigma) > 0$ . By (11), (12) and definition of  $U_\alpha$  we obtain the inequality

$$\left\| \int_\sigma^{\sigma+\eta} [f(y(t), t) - f(x(t), t)] dt \right\| \leq U_\alpha(\sigma + \eta) - U_\alpha(\sigma)$$

for  $\sigma \in [t_0, t_1], \eta \in [0, \delta(\sigma)]$ , then by Theorem 1.3 and Theorem 1.17 in [5], we have  $\Phi(\left\| \int_\sigma^{\sigma+\eta} [f(y(t), t) - f(x(t), t)] dt \right\|) \leq \Phi(U_\alpha(\sigma + \eta) - U_\alpha(\sigma)) = V_\Phi(U_\alpha; [\sigma, \sigma + \eta]) \leq V_\Phi(U_\alpha; [t_0, \sigma + \eta]) - V_\Phi(U_\alpha; [t_0, \sigma])$ , therefor by (4) we have

$$\begin{aligned} &V(\sigma + \eta, y(\sigma + \eta)) - V(\sigma, x(\sigma)) \\ &\leq N(V_\Phi(\Psi; [t_0, \sigma + \eta]) - V_\Phi(\Psi; [t_0, \sigma]) + V_\Phi(U_\alpha; [t_0, \sigma + \eta]) - V_\Phi(U_\alpha; [t_0, \sigma])) + \eta R + \eta \varepsilon \\ &= G(\sigma + \eta) - G(\sigma), \end{aligned} \quad (15)$$

for  $\sigma \in [t_0, t_1], \eta \in [0, \delta(\sigma)]$  where  $G(t) = NV_\Phi(\Psi; [t_0, t]) + R(t - t_0) + \varepsilon(t - t_0) + NV_\Phi(U_\alpha; [t_0, t])$ ,  $G$  is of bounded  $\Phi$ -variation on  $[t_0, t_1]$  and continuous from the left on  $(t_0, t_1]$ . By (14), (15) and Lemma 2.2, we obtain

$$\begin{aligned} &V(t_1, y(t_1)) - V(t_0, y(t_0)) \leq G(t_1) - G(t_0) \\ &= NV_\Phi(\Psi; [t_0, t_1]) + R(t_1 - t_0) + \varepsilon(t_1 - t_0) + NV_\Phi(U_\alpha; [t_0, t_1]) \\ &< NV_\Phi(\Psi; [t_0, t_1]) + R(t_1 - t_0) + \varepsilon(t_1 - t_0) + N\Phi\left(\frac{\varepsilon}{N}\right), \end{aligned}$$

Since  $\varepsilon$  can be arbitrary small, we obtain the result given in (3) from this inequality.

**Theorem 3.2.** Assume that  $V : [0, +\infty) \times \bar{B}_a \rightarrow R, 0 < a < c$  is such that for every  $x \in \bar{B}_a = \{y \in R^n : \|y\| \leq a\}$  the function  $V(\cdot, x)$  is continuous from the left and

(1) Assume that the function  $V(t, x)$  is positive definite, i.e. there exists a continuous and increasing real function  $v : [0, +\infty) \rightarrow R$  Such that  $v(\rho) = 0 \iff \rho = 0$ ;

(2) for all  $(t, x) \in [0, +\infty) \times \bar{B}_a$ , we have

$$V(t, x) \geq v(\Phi(\|x\|)) \quad (16)$$

and

$$V(t, 0) = 0. \quad (17)$$

For any  $x, y \in \bar{B}_a$  and  $L > 0$  is a constant, we have

$$|V(t, x) - V(t, y)| \leq L\Phi(\|x - y\|). \quad (18)$$

If the function  $V(t, x(t))$  is nonincreasing and of bounded  $\Phi$ -variation along every solution  $x$  of (\*) then the solution  $x \equiv 0$  of (\*) is  $\Phi$ -variationally stable.

**Proof.** Since we assume that the function  $V(t, x(t))$  is nonincrease and of bounded  $\Phi$ -variation whenever  $x : [\alpha, \beta] \rightarrow R^n$  is a solution of (\*), we have

$$\lim_{\eta \rightarrow 0^+} \sup \frac{V(t + \eta, x(t + \eta)) - V(t, x(t))}{\eta} \leq 0 \quad (19)$$

for  $t \in [\alpha, \beta]$ .

Let  $\varepsilon > 0$  be given and let  $y : [t_0, t_1] \rightarrow R^n$ ,  $0 \leq t_0 < t_1 < +\infty$  be of bounded  $\Phi$ -variation on  $[t_0, t_1]$  and continuous from the left in  $(t_0, t_1]$ . Since the function  $V$  satisfies the assumptions (2) of Theorem 3.1 with  $H \equiv 0$ , We obtain by (3), (17) and (18) the inequality

$$\begin{aligned} V(r, y(r)) &\leq V(t_0, y(t_0)) + NV_{\Phi}((y(s) - \int_{t_0}^s f(y(t), t)dt); [t_0, r]) \\ &\leq L\Phi(\|y(t_0)\|) + NV_{\Phi}((y(s) - \int_{t_0}^s f(y(t), t)dt); [t_0, r]) \\ &\leq N\Phi(\|y(t_0)\|) + NV_{\Phi}((y(s) - \int_{t_0}^s f(y(t), t)dt); [t_0, r]) \end{aligned} \quad (20)$$

holds for every  $r \in [t_0, t_1]$ .

Define  $\alpha(\varepsilon) = \inf_{r \leq \varepsilon} v(r)$ , then  $\alpha(\varepsilon) > 0$  for  $\varepsilon > 0$  and  $\lim_{\varepsilon \rightarrow 0^+} \alpha(\varepsilon) = 0$ . Further choose  $\delta(\varepsilon) > 0$  such that  $2N\delta(\varepsilon) < \alpha(\varepsilon)$ . If function  $y$  satisfying  $\Phi(\|y(t_0)\|) < \delta(\varepsilon)$  and  $V_{\Phi}((y(s) - \int_{t_0}^s f(y(t), t)dt); [t_0, r]) < \delta(\varepsilon)$ , then by (20), we have

$$V(r, y(r)) \leq 2N\delta(\varepsilon) < \alpha(\varepsilon), r \in [t_0, t_1]. \quad (21)$$

If there exists a  $\tilde{t} \in [t_0, t_1]$  such that  $\Phi(\|y(\tilde{t})\|) \geq \varepsilon$  and by (16) we get the inequality  $V(\tilde{t}, y(\tilde{t})) \geq v(\Phi(\|y(\tilde{t})\|)) \geq \inf_{r \leq \varepsilon} v(r) = \alpha(\varepsilon)$ , which contradicts (21). Hence  $\Phi(\|y(t)\|) < \varepsilon$  for all  $t \in (t_0, t_1]$  and by Definition 2.7 the solution  $x \equiv 0$  of (\*) is  $\Phi$ -variationally stable.

**Theorem 3.3.** Let  $V : [0, +\infty) \times \bar{B}_a \rightarrow R$  be a function with the properties given in Theorem 3.2. If for every solution  $x : [t_0, t_1] \rightarrow \bar{B}_a$  of bounded  $\Phi$ -variation of (\*) the inequality

$$\lim_{\eta \rightarrow 0^+} \sup \frac{V(t + \eta, x(t + \eta)) - V(t, x(t))}{\eta} \leq -H(x(t)) \quad (22)$$

holds for every  $t \in [t_0, t_1]$ , where  $H : R^n \rightarrow R$  is continuous,  $H(0) = 0, H(x) > 0$  for  $x \neq 0$ , then the solution  $x \equiv 0$  of (\*) is  $\Phi$ -variationally-asymptotically stable.

**Proof.** From (22) it is clear that the function  $V(t, x(t))$  is of bounded  $\Phi$ -variation and nonincreasing along every solution  $x(t)$  of  $(*)$  and therefore by Theorem 3.2, the solution  $x \equiv 0$  is  $\Phi$ -variationally stable.

It remains to show that the solution  $x \equiv 0$  is  $\Phi$ -variationally attracting. From the  $\Phi$ -variational stability of the solution  $x \equiv 0$  of  $(*)$ , there is a  $\delta_0 \in (0, \Phi(a))$  such that if  $y : [t_0, t_1] \rightarrow R^n$  is of bounded  $\Phi$ -variation on  $[t_0, t_1]$  where  $0 \leq t_0 < t_1 < +\infty$ ,  $y$  is continuous from the left on  $(t_0, t_1]$  and such that  $\Phi(\|y(t_0)\|) < \delta_0$  and  $V_\Phi((y(s) - \int_{t_0}^s f(y(t), t)dt); [t_0, t_1]) < \delta_0$ , then  $\Phi(\|y(t)\|) < \Phi(a)$  for  $t \in [t_0, t_1]$ , by the definition of  $\Phi(u)$ , we have  $\|y(t)\| < a$ , i.e for every  $t \in (t_0, t_1]$  we obtain  $y(t) \in \bar{B}_a$ . Let  $\varepsilon > 0$  be arbitrary, from the  $\Phi$ -variational stability of the solution  $x \equiv 0$  we obtain that there is a  $\delta(\varepsilon) > 0$  such that for every  $y : [t_0, t_1] \rightarrow R^n$  of bounded  $\Phi$ -variation on  $[t_0, t_1]$  where  $0 \leq t_0 < t_1 < +\infty$ ,  $y$  is continuous from the left on  $(t_0, t_1]$  and such that  $\Phi(\|y(t_0)\|) < \delta(\varepsilon)$  and  $V_\Phi((y(s) - \int_{t_0}^s f(y(t), t)dt); [t_0, t_1]) < \delta(\varepsilon)$ , we have  $\Phi(\|y(t)\|) < \varepsilon$  for  $t \in [t_0, t_1]$ . Define  $\gamma(\varepsilon) = \min(\delta_0, \delta(\varepsilon))$  and  $T(\varepsilon) = -N \frac{\delta_0 + \gamma(\varepsilon)}{R} > 0$  where  $R = \sup\{-H(x) : \gamma(\varepsilon) \leq \Phi(\|x\|) < \varepsilon\} = -\inf\{H(x) : \gamma(\varepsilon) \leq \Phi(\|x\|) < \varepsilon\} < 0$  and assume that  $y : [t_0, t_1] \rightarrow R^n$  of bounded  $\Phi$ -variation on  $[t_0, t_1]$  where  $0 \leq t_0 < t_1 < +\infty$ ,  $y$  is continuous from the left on  $(t_0, t_1]$  and such that  $\Phi(\|y(t_0)\|) < \delta_0$  and

$$V_\Phi((y(s) - \int_{t_0}^s f(y(t), t)dt); [t_0, t_1]) < \gamma(\varepsilon). \quad (23)$$

Assume that  $T(\varepsilon) < t_1 - t_0$ , i.e.  $t_0 + T(\varepsilon) < t_1$ . We show that there exists a  $t^* \in [t_0, t_0 + T(\varepsilon)]$  such that  $\Phi(\|y(t^*)\|) < \gamma(\varepsilon)$ . Assume the contrary, i.e.  $\Phi(\|y(s)\|) \geq \gamma(\varepsilon)$  for every  $s \in [t_0, t_0 + T(\varepsilon)]$ . Theorem 3.1 yields

$$\begin{aligned} & V(t_0 + T(\varepsilon), y(t_0 + T(\varepsilon))) - V(t_0, y(t_0)) \\ & \leq NV_\Phi((y(s) - \int_{t_0}^s f(y(t), t)dt); [t_0, t_0 + T(\varepsilon)]) + RT(\varepsilon) \\ & < N\gamma(\varepsilon) + R \frac{-N(\delta_0 + \gamma(\varepsilon))}{R} = -N\delta_0. \end{aligned}$$

Hence  $V(t_0 + T(\varepsilon), y(t_0 + T(\varepsilon))) \leq V(t_0, y(t_0)) - N\delta_0 \leq L\Phi(\|y(t_0)\|) - N\delta_0 \leq N\Phi(\|y(t_0)\|) - N\delta_0 < (N - N)\delta_0 = 0$ , and this contradicts the inequality  $V(t_0 + T(\varepsilon), y(t_0 + T(\varepsilon))) \geq v(\Phi(\|y(t_0 + T(\varepsilon))\|)) \geq v(\gamma(\varepsilon)) > 0$ . Hence necessarily there is a  $t^* \in [t_0, t_0 + T(\varepsilon)]$  such that  $\Phi(\|y(t^*)\|) < \gamma(\varepsilon)$ . And by (23), we have  $\Phi(\|y(t)\|) < \varepsilon$  for  $t \in [t^*, t_1]$ . Consequently,  $\Phi(\|y(t)\|) < \varepsilon$  for  $t > t_0 + T(\varepsilon)$ . Therefore, the solution  $x \equiv 0$  is a  $\Phi$ -variationally attracting solution of  $(*)$ .

**Remark 3.1.** If the function  $\Phi(u)$  is defined in last section such that  $0 < \frac{\Phi(u)}{u} < +\infty$  then by Theorem 1.15 in paper [5], we have  $BV_\Phi[\alpha, \beta] = BV[\alpha, \beta]$  where  $BV_\Phi[\alpha, \beta]$  and  $BV[\alpha, \beta]$  denote the classes of the functions bounded  $\Phi$ -variation and the functions bounded variation on  $[\alpha, \beta]$  respectively in usual sense. Hence, the main results of this paper are equivalent to the results of variational stability in [4].

If  $\lim_{u \rightarrow 0^+} \frac{\Phi(u)}{u} = 0$  then by Theorem 1.15 in [5], we have  $BV[\alpha, \beta] \subset BV_\Phi[\alpha, \beta]$ . Such as if  $\Phi(u) = u^p$  ( $1 < p < +\infty$ ) we have  $\lim_{u \rightarrow 0^+} \frac{\Phi(u)}{u} = \frac{u^p}{u} = 0$ . Therefore these results are essential generalization of variation stability of bounded variation solution to discontinuous system in [4].



## References

- [1] J. He and P. Chen, Some aspects of the theory and applications of discontinuous differential equations [J], Adv. in Math., **16**(1987), 17-32.
- [2] A. F. Filippov, Differential equations with discontinuous right hand side [J], Math, USSR-sb, **51**(1960).
- [3] Cunxin Wu and Baolin Li, Bounded variation solutions for discontinuous systems [J], Journal of Mathematics Study, **31**(1998), No. 4, 417-427.
- [4] Baolin Li and Quande Shang, Variational stability for a class of discontinuous system [J], Journal of Northwest Normal University (Natural Science), **42**(2006), No. 2, 1-3.
- [5] J. Musielak and W. Orlicz, On generalized variation(I) [J], Studia Math., **18**(1959), 11-41.
- [6] R. Lesniewz and W. Orlicz, On generalized variation(II) [J], Studia Math., **45**(1973), 71-109.
- [7] Yanping Xiao and Baolin Li, Bounded  $\Phi$ -variation solutions for a class of discontinuous system [J], Chinese Journal of engineering mathematics, **25**(2008), No. 3, 489-494.
- [8] S. Schwabik, Generalized ordinary differential equations [M], Singapore: World Scientific, 1992.

# Note on right circulant matrices with Pell and Pell-Lucas sequences

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**Abstract** In this paper, we shall construct right circulant matrices from the Pell and Pell-Lucas sequences. Furthermore, we derive the eigenvalues and Euclidean norm of these matrices. We also give some bounds on their spectral norms.

**Keywords** Determinant, eigenvalue, Euclidean norm, Pell sequence, Pell-Lucas sequence, right circulant matrix, spectral norm.

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## §1. Introduction

The Pell and Pell-Lucas sequence satisfy the following recurrence relations, respectively:

$$P_n = 2P_{n-1} + P_{n-2}, P_0 = 0, P_1 = 1;$$

$$Q_n = 2Q_{n-1} + Q_{n-2}, Q_0 = Q_1 = 2.$$

Their  $n^{th}$  are given by

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}, \quad (1)$$

$$Q_n = \alpha^n + \beta^n. \quad (2)$$

where  $\alpha = 1 + \sqrt{2}$  and  $\beta = 1 - \sqrt{2}$ .

Bozkurt <sup>[1]</sup> constructed the right circulant matrices with Pell and Pell-Lucas sequences. These matrices take the following forms, respectively:

$$RCIRC_n(\vec{P}) = \begin{pmatrix} P_0 & P_1 & P_2 & \dots & P_{n-2} & P_{n-1} \\ P_{n-1} & P_0 & P_1 & \dots & P_{n-3} & P_{n-2} \\ P_{n-2} & P_{n-1} & P_0 & \dots & P_{n-4} & P_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ P_2 & P_3 & P_4 & \dots & P_0 & P_1 \\ P_1 & P_2 & P_3 & \dots & P_{n-1} & P_0 \end{pmatrix} \quad (3)$$

$$RCIRC_n(\vec{Q}) = \begin{pmatrix} Q_0 & Q_1 & Q_2 & \dots & Q_{n-2} & Q_{n-1} \\ Q_{n-1} & Q_0 & Q_1 & \dots & Q_{n-3} & Q_{n-2} \\ Q_{n-2} & Q_{n-1} & Q_0 & \dots & Q_{n-4} & Q_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_2 & Q_3 & Q_4 & \dots & Q_0 & Q_1 \\ Q_1 & Q_2 & Q_3 & \dots & Q_{n-1} & Q_0 \end{pmatrix} \quad (4)$$

He presented the determinants and the inverses of these matrices.

## §2. Preliminary results

The following will be used to prove some of the main results.

**Lemma 2.1.**

$$\sum_{k=0}^{n-1} [\alpha^k - \beta^k] \omega^{-mk} = \frac{\alpha^n - \beta^n + (\alpha^{n-1} - \beta^{n-1})\omega^{-m} - (\alpha - \beta)}{\omega^{-2m} + 2\omega^{-m} - 1}, \quad (5)$$

where  $\omega = e^{2\pi i/n}$ .

**Proof.**

$$\begin{aligned} \sum_{k=0}^{n-1} [\alpha^k - \beta^k] \omega^{-mk} &= \frac{1 - \alpha^n}{1 - \alpha\omega^{-m}} + \frac{1 - \beta^n}{1 - \beta\omega^{-m}} \\ &= \frac{(1 - \alpha^n)(1 - \beta\omega^{-m}) - (1 - \beta^n)(1 - \alpha\omega^{-m})}{1 - 2\omega^{-m} - \omega^{-2m}} \\ &= \frac{\alpha^n - \beta^n + (\alpha^{n-1} - \beta^{n-1})\omega^{-m} - (\alpha - \beta)}{\omega^{-2m} + 2\omega^{-m} - 1}. \end{aligned}$$

**Lemma 2.2.**

$$\sum_{k=0}^{n-1} [\alpha^k - \beta^k]^2 = \frac{Q_{2n} + Q_{2n-2}}{4} - (-1)^n. \quad (6)$$

**Proof.**

$$\begin{aligned} \sum_{k=0}^{n-1} [\alpha^k - \beta^k]^2 &= \sum_{k=0}^{n-1} [\alpha^{2k} - 2(\alpha\beta)^k + \beta^{2k}] \\ &= \frac{1 - \alpha^{2n}}{1 - \alpha^2} + \frac{1 - \beta^{2n}}{1 - \beta^2} - 2 \left( \frac{1 - (-1)^n}{2} \right) \\ &= \frac{(1 - \alpha^{2n})(1 - \beta^2) + (1 - \beta^{2n})(1 - \alpha^2)}{(1 - \alpha^2)(1 - \beta^2)} + (-1)^n - 1 \\ &= \frac{-4 - (\alpha^{2n} + \beta^{2n}) - (\alpha^{2n-2} + \beta^{2n-2})}{-4} + (-1)^n - 1 \\ &= \frac{Q_{2n} + Q_{2n-2}}{4} - (-1)^n. \end{aligned}$$

**Lemma 2.3.**

$$\sum_{k=0}^{n-1} [\alpha^k + \beta^k]^2 = \frac{Q_{2n} + Q_{2n-2}}{4} + 2 - (-1)^n. \quad (7)$$

**Proof.**

$$\begin{aligned}
\sum_{k=0}^{n-1} [\alpha^k + \beta^k]^2 &= \sum_{k=0}^{n-1} [\alpha^{2k} + 2(\alpha\beta)^k + \beta^{2k}] \\
&= \frac{1 - \alpha^{2n}}{1 - \alpha^2} + \frac{1 - \beta^{2n}}{1 - \beta^2} + 2 \left( \frac{1 - (-1)^n}{2} \right) \\
&= \frac{(1 - \alpha^{2n})(1 - \beta^2) + (1 - \beta^{2n})(1 - \alpha^2)}{(1 - \alpha^2)(1 - \beta^2)} + 1 - (-1)^n \\
&= \frac{-4 - (\alpha^{2n} + \beta^{2n}) - (\alpha^{2n-2} + \beta^{2n-2})}{-4} + 1 - (-1)^n \\
&= \frac{Q_{2n} + Q_{2n-2}}{4} + 2 - (-1)^n.
\end{aligned}$$

For the rest of the paper, we will use  $|A|$ ,  $\|A\|_E$  and  $\|A\|_2$  to denote the determinant, euclidean norm and spectral norm of matrix  $A$ , respectively.

### §3. Main results

**Theorem 3.1.** The eigenvalues of  $RCIRC_n(\vec{P})$  are given by

$$\lambda_m = \frac{(P_{n-1} - 1)\omega^{-m} + P_n}{\omega^{-2m} + 2\omega^{-m} - 1}. \quad (8)$$

**Proof.**

$$\begin{aligned}
\lambda_m &= \sum_{k=0}^{n-1} P_k \omega^{-mk} \\
&= \sum_{k=0}^{n-1} \frac{[\alpha^k - \beta^k] \omega^{-mk}}{2\sqrt{2}} \\
&= \frac{\alpha^n - \beta^n + (\alpha^{n-1} - \beta^{n-1})\omega^{-m} - (\alpha - \beta)}{2\sqrt{2}(\omega^{-2m} + 2\omega^{-m} - 1)} \\
&= \frac{(P_{n-1} - 1)\omega^{-m} + P_n}{\omega^{-2m} + 2\omega^{-m} - 1}.
\end{aligned}$$

**Theorem 3.2.** The eigenvalues of  $RCIRC_n(\vec{Q})$  are given by

$$\mu_m = \frac{Q_{n+1} + 2(\omega^{-m} - 1)}{\omega^{-2m} + 2\omega^{-m} - 1}. \quad (9)$$

**Proof.**

$$\begin{aligned}
\mu_m &= \sum_{k=0}^{n-1} Q_k \omega^{-mk} \\
&= \sum_{k=0}^{n-1} [\alpha^k + \beta^k] \omega^{-mk} \\
&= \frac{1 - \alpha^n}{1 - \alpha \omega^{-m}} + \frac{1 - \beta^n}{1 - \beta \omega^{-m}} \\
&= \frac{(1 - \alpha^n)(1 - \beta \omega^{-m}) + (1 - \beta^n)(1 - \alpha \omega^{-m})}{1 - 2\omega^{-m} - \omega^{-2m}} \\
&= \frac{2 - Q_n - 2\omega^{-m} - Q_{n-1}\omega^{-m}}{1 - 2\omega^{-m} - \omega^{-2m}} \\
&= \frac{Q_{n+1} + 2(\omega^{-m} - 1)}{\omega^{-2m} + 2\omega^{-m} - 1}.
\end{aligned}$$

**Theorem 3.3.**

$$\left\| RCIRC_n(\vec{P}) \right\|_E = \frac{\sqrt{2n(Q_{2n} + Q_{2n-2}) - (-1)^n 64}}{8}. \quad (10)$$

**Proof.**

$$\begin{aligned}
\left\| RCIRC_n(\vec{P}) \right\|_E &= \sqrt{n \sum_{k=0}^{n-1} \left[ \frac{\alpha^k - \beta^k}{2\sqrt{2}} \right]^2} \\
&= \sqrt{\frac{n}{8} \sum_{k=0}^{n-1} [\alpha^k - \beta^k]^2} \\
&= \sqrt{\frac{n}{8} \left[ \frac{Q_{2n} + Q_{2n-2}}{4} - (-1)^n \right]} \\
&= \frac{\sqrt{2n(Q_{2n} + Q_{2n-2}) - (-1)^n 64}}{8}.
\end{aligned}$$

**Theorem 3.4.**

$$\left\| RCIRC_n(\vec{Q}) \right\|_E = \frac{\sqrt{n(Q_{2n} + Q_{2n-2}) + 8 - (-1)^n 4}}{2}. \quad (11)$$

**Proof.**

$$\begin{aligned}
\left\| RCIRC_n(\vec{Q}) \right\|_E &= \sqrt{n \sum_{k=0}^{n-1} [\alpha^k + \beta^k]^2} \\
&= \sqrt{n \left[ \frac{Q_{2n} + Q_{2n-2}}{4} + 2 - (-1)^n \right]} \\
&= \frac{\sqrt{n(Q_{2n} + Q_{2n-2}) + 8 - (-1)^n 4}}{2}.
\end{aligned}$$

**Theorem 3.5.**

$$\frac{P_{n-1} - P_n - 1}{4} \leq \left\| RCIRC_n(\vec{P}) \right\|_2 \leq \frac{P_{n+1} - 1}{2}. \quad (12)$$

**Proof.** Note that

$$\left\| RCIRC_n(\vec{P}) \right\|_2 = \max \{ |\lambda_m| \},$$

hence, we have the following inequality

$$\begin{aligned} \frac{|(P_{n-1} - 1)\omega^{-m}| - P_n}{|\omega^{-2m} + 2\omega^{-m} - 1|} &\leq \left\| RCIRC_n(\vec{P}) \right\|_2 \leq \frac{|(P_{n-1} - 1)\omega^{-m}| + P_n}{|\omega^{-2m} + 2\omega^{-m} - 1|} \\ \frac{P_{n-1} - P_n - 1}{4} &\leq \left\| RCIRC_n(\vec{P}) \right\|_2 \leq \frac{P_{n-1} - 1 + P_n}{2} \\ \frac{P_{n-1} - P_n - 1}{4} &\leq \left\| RCIRC_n(\vec{P}) \right\|_2 \leq \frac{P_{n+1} - 1}{2}. \end{aligned}$$

**Theorem 3.6.**

$$\frac{Q_{n-1}}{4} \leq \left\| RCIRC_n(\vec{Q}) \right\|_2 \leq \frac{Q_{n+1} + 4}{2}. \quad (13)$$

**Proof.** Similar as the previous theorem 3.5.

## References

- [1] D. Bozkurt, On the determinants and inverses of circulant matrices with Pell and Pell-Lucas numbers, arXiv: 1201, 6061v1 [math. NA], 29 Jan, 2012.

# Evaluation of a summation formula confederated with contiguous relation

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**Abstract** The main aim of the present paper is to evaluate a summation formula confederated with contiguous relation and recurrence relation.

**Keywords** Gaussian hypergeometric function, contiguous function, recurrence relation, Bailey summation theorem and Legendre duplication formula.

**2010 Mathematics Subject Classification:** 33C60, 33C70, 33D15, 33D50, 33D60.

## §1. Introduction and preliminaries

**Definition 1.1.** Generalized Gaussian hypergeometric function of one variable is defined by

$${}_A F_B \left[ \begin{matrix} a_1, a_2, \dots, a_A & ; \\ & z \end{matrix} \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_A)_k z^k}{(b_1)_k (b_2)_k \dots (b_B)_k k!},$$

or

$${}_A F_B \left[ \begin{matrix} (a_A) & ; \\ (b_B) & ; \end{matrix} \right] z \equiv {}_A F_B \left[ \begin{matrix} (a_j)_{j=1}^A & ; \\ (b_j)_{j=1}^B & ; \end{matrix} \right] z = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!}, \quad (1)$$

where the parameters  $b_1, b_2, \dots, b_B$  are neither zero nor negative integers and  $A, B$  are non-negative integers.

**Definition 1.2.** Contiguous relation [E. D. p. 51(10), Andrews p. 363(9. 16)] is defined as follows:

$$(a-b) {}_2F_1 \left[ \begin{matrix} a, b & ; \\ c & ; \end{matrix} \right] z = a {}_2F_1 \left[ \begin{matrix} a+1, b & ; \\ c & ; \end{matrix} \right] z - b {}_2F_1 \left[ \begin{matrix} a, b+1 & ; \\ c & ; \end{matrix} \right] z. \quad (2)$$

**Definition 1.3.** Recurrence relation of gamma function is defined as follows:

$$\Gamma(z+1) = z \Gamma(z). \quad (3)$$

**Definition 1.4.** Legendre duplication formula [ Bells & Wong p. 26(2.3.1) ] is defined as follows:

$$\sqrt{\pi} \Gamma(2z) = 2^{(2z-1)} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right), \quad (4)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \frac{2^{(b-1)} \Gamma(\frac{b}{2}) \Gamma(\frac{b+1}{2})}{\Gamma(b)} \quad (5)$$

$$= \frac{2^{(a-1)} \Gamma(\frac{a}{2}) \Gamma(\frac{a+1}{2})}{\Gamma(a)}. \quad (6)$$

**Definition 1.5.** Bailey summation theorem [Prud, p. 491(7.3.7.8)] is defined as follows:

$${}_2F_1 \left[ \begin{matrix} a, 1-a & ; & 1 \\ c & ; & \frac{1}{2} \end{matrix} \right] = \frac{\Gamma(\frac{c}{2}) \Gamma(\frac{c+1}{2})}{\Gamma(\frac{c+a}{2}) \Gamma(\frac{c+1-a}{2})} = \frac{\sqrt{\pi} \Gamma(c)}{2^{c-1} \Gamma(\frac{c+a}{2}) \Gamma(\frac{c+1-a}{2})}. \quad (7)$$

## §2. Main results of summation formulae

$$\begin{aligned} & {}_2F_1 \left[ \begin{matrix} a, -a-49 & ; & 1 \\ c & ; & \frac{1}{2} \end{matrix} \right] \\ &= \frac{\sqrt{\pi} \Gamma(c)}{2^{c+49}} \left[ \frac{40015948329812641136799430213632000000}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \right. \\ & \quad + \frac{-74697042588755634816870901715435520000a}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\ & \quad + \frac{45701551565700312458589664488136704000a^2}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\ & \quad + \frac{-12557326585358016296749495447708262400a^3}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\ & \quad + \frac{1607017301962727354472466577244395520a^4}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\ & \quad + \frac{-66334092838839671097859298211598848a^5 - 4173963485546119223032377440607360a^6}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\ & \quad + \frac{354268334898336692529886398612096a^7 + 6493428631831030022245150741472a^8}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\ & \quad + \frac{-716075598615688398506581518048a^9 - 13409906223325457338650846040a^{10}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\ & \quad + \frac{695228979926866432431858936a^{11} + 20318017073982774702824162a^{12}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\ & \quad + \frac{-181081529755913127436368a^{13} - 12758198887309415982820a^{14}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \end{aligned}$$



$$\begin{aligned}
& + \frac{-113809757878112295384a^{15} + 1948115287787037182a^{16} + 46874133673870272a^{17}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{257645168805200a^{18} - 1896116954424a^{19} - 31420958338a^{20} - 136837008a^{21}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-32980a^{22} + 1176a^{23} + 2a^{24} + 102873723702221439014473177529057280000c}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-143953054337702235151044038042320896000ac}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{69785186656677903329815708782664089600a^2c}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-15254087824286337343500853016781127680a^3c}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{1470184367841770478008747704464310272a^4c}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-25186819706236967969752199080869888a^5c - 4916413813362186868890467613499392a^6c}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{180523758084624041877205846241280a^7c + 9444650003452625898653645586432a^8c}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-290073752007187728910991044608a^9c - 13790932677896490825858473472a^{10}c}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{132168254665462941656989440a^{11}c + 11478210451803468174127872a^{12}c}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{82419396059301592042752a^{13}c - 3441243829617943295232a^{14}c}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-68832268710004800000a^{15}c - 120844174461424128a^{16}c + 8743439744911872a^{17}c}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{96443587688448a^{18}c + 222439176960a^{19}c - 2314760448a^{20}c - 16144128a^{21}c}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-29952a^{22}c + 107759352272213925707820568758190080000c^2}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-119604791694733248989672753097867264000ac^2}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{46915581673954062917864644291866132480a^2c^2}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-8179716567466648414966719773717397504a^3c^2}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{572414151475230486442581108591747072a^4c^2}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{4225122815762079559693237812774912a^5c^2 - 2107743828904816275105349039067648a^6c^2}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{21777324127000959865037942782208a^7c^2 + 3811313237604914006353415570432a^8c^2}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-17208144178264485968744984832a^9c^2 - 4072656512763355130085794976a^{10}c^2}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-30448585606100284800091824a^{11}c^2 + 1971285851965809130849552a^{12}c^2}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{37909392941438882335472a^{13}c^2 - 136814112867707752048a^{14}c^2}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-10018332037253860320a^{15}c^2 - 86695521821402208a^{16}c^2 + 228479601333984a^{17}c^2}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{7725554252416a^{18}c^2 + 40796996240a^{19}c^2 + 25577552a^{20}c^2 - 336336a^{21}c^2}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-624a^{22}c^2 + 64423398177963803144998543335358464000c^3}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-58300150202899337016565241575133675520ac^3}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{18691960982318475085206332143677997056a^2c^3}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-2586065877925151796735934133731590144a^3c^3}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{122171345804374971207330799098003456a^4c^3}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{4533540855919510991454734335180800a^5c^3 - 467631969431579769690069597929472a^6c^3}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-4697327171517980882726705283072a^7c^3 + 722969542181022180843132002304a^8c^3}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{9083202730777950404000716800a^9c^3 - 529578571988520511523748864a^{10}c^3}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-11402466632175022106486784a^{11}c^3 + 108169499556766290779136a^{12}c^3}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{5063222588621911633920a^{13}c^3 + 32000805092958873600a^{14}c^3 - 498995580905902080a^{15}c^3}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-8205717988435968a^{16}c^3 - 29314507622400a^{17}c^3 + 167330042880a^{18}c^3 + 1506785280a^{19}c^3}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{3075072a^{20}c^3 + 25347138973058359260380026128826368000c^4}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{-18978543797716490715225178573727858688ac^4}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{4991684025479138391935744923474722816a^2c^4}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-541099749641591095487701589399519232a^3c^4}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{14606689411535623104290723942940672a^4c^4}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{1292549598384288914173152912040960a^5c^4 - 59385691383614919077242368788992a^6c^4}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-1892451113166353101369696786944a^7c^4 + 72169632318929524275249528576a^8c^4}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{2228687725107369556056237120a^9c^4 - 28472710924167755430485664a^{10}c^4}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-1409430360340354713769408a^{11}c^4 - 5501770617352422372960a^{12}c^4}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{297674907085096661120a^{13}c^4 + 3995332631611006400a^{14}c^4 + 1153916414851200a^{15}c^4}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-295200075538368a^{16}c^4 - 1932687556800a^{17}c^4 - 1947065120a^{18}c^4 + 15695680a^{19}c^4}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{32032a^{20}c^4 + 7099535118854336323202874869003845632c^5}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-4432336230110420896646668265044574208ac^5}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{955466413384038667615798431427067904a^2c^5}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-79380397141399923587473758714593280a^3c^5 + 643805931621664284893495124295680a^4c^5}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{209465328810891152290757390991360a^5c^5 - 4028319000276505780447170035712a^6c^5}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-290915966729411309455672688640a^7c^5 + 3124145533111366614800302080a^8c^5}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{241803413658639276317153280a^9c^5 + 584445378775657625081856a^{10}c^5}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-87844798556732071895040a^{11}c^5 - 1067805772051345367040a^{12}c^5}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{6128657556733071360a^{13}c^5 + 189548697075978240a^{14}c^5 + 958658985123840a^{15}c^5}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{-3202625986560a^{16}c^5 - 40683202560a^{17}c^5 - 92252160a^{18}c^5}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{1487119300880904242605217024921567232c^6 - 776777553617389082534820001691467776ac^6}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{136593392051968205997829269713846272a^2c^6 - 8388092183067964659019677414162432a^3c^6}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-91340670507632683968759660462080a^4c^6 + 22224973282782547275305692944384a^5c^6}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-35529674554603169040172140544a^6c^6 - 26210702433049018598332609536a^7c^6}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-103200723777727230528706560a^8c^6 + 14895636537534753758389632a^9c^6}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{182403094584283413160320a^{10}c^6 - 2663633385875852662272a^{11}c^6}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-62256983715951664640a^{12}c^6 - 174817816321564416a^{13}c^6 + 4084756093786368a^{14}c^6}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{34430420344320a^{15}c^6 + 48609200640a^{16}c^6 - 282522240a^{17}c^6 - 640640a^{18}c^6}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{240838218499636521398013613710508032c^7 - 105287696394810237973591256098406400ac^7}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{14976654351475036084045130810523648a^2c^7 - 638818690035905266292950472589312a^3c^7}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-2096054159250061772244448268288a^4c^7 + 1615109961509085989321865953280a^5c^7}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{21023698439479542356321501184a^6c^7 - 1497276217131824539913158656a^7c^7}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-23179379774042918881787904a^8c^7 + 515692831831999968706560a^9c^7}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{12163046240440205180928a^{10}c^7 - 6024819137364099072a^{11}c^7 - 1837489800964866048a^{12}c^7}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-12946877731307520a^{13}c^7 + 23007583272960a^{14}c^7 + 495947612160a^{15}c^7 + 1265172480a^{16}c^7}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{30871602486909275910830889875013632c^8 - 11272571429148914722533062085705728ac^8}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{1280968945576182794260760676794368a^2c^8 - 33589734750958810846109982916608a^3c^8}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-2206717841266713417081546031104a^4c^8 + 79728366271118889239432724480a^5c^8}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{2306069297890733863547265024a^6c^8 - 52263325577453158896279552a^7c^8}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-1561668419447767019271168a^8c^8 + 6335745740330385899520a^9c^8}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{433099863114577072128a^{10}c^8 + 2445053402814038016a^{11}c^8 - 26209896905127936a^{12}c^8}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-303502718238720a^{13}c^8 - 568422666240a^{14}c^8 + 2583060480a^{15}c^8 + 6589440a^{16}c^8}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{3185642085289031342835468884508672c^9 - 967550638958541171077959970193408ac^9}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{86299774734978829551171583082496a^2c^9 - 998945521658261204553034629120a^3c^9}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-153617690085710615493130321920a^4c^9 + 2411197000642734061356318720a^5c^9}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{137452472767000072564506624a^6c^9 - 785064641247377381130240a^7c^9}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-61009257199589456609280a^8c^9 - 256032809547607572480a^9c^9}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{8769591108516446208a^{10}c^9 + 88785671187333120a^{11}c^9 - 46103447470080a^{12}c^9}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-3278764769280a^{13}c^9 - 9559080960a^{14}c^9 + 267889258438085483657974117302272c^{10}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-67263084939581397145810722881536ac^{10} + 4593733502680143944975661400064a^2c^{10}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{8539339140610043749157830656a^3c^{10} - 7676662187554452553435054080a^4c^{10}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{19441815522855857897013248a^5c^{10} + 5387422458233145326796800a^6c^{10}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{20474586985880613634048a^7c^{10} - 1490183150114131066880a^8c^{10}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-14543250526116888576a^9c^{10} + 83327898738180096a^{10}c^{10} + 1503081400877056a^{11}c^{10}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{3662123745280a^{12}c^{10} - 13661519872a^{13}c^{10} - 39829504a^{14}c^{10}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{18518169236138399504184033411072c^{11} - 3812360305782072606527229788160ac^{11}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{192340688684970209357454114816a^2c^{11} + 2855197808016937994321657856a^3c^{11}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{-281873037311025074002722816a^4c^{11} - 2077996489024813179863040a^5c^{11}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{143506954311464043675648a^6c^{11} + 1541710276583120437248a^7c^{11}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-21366367851509710848a^8c^{11} - 341036842572840960a^9c^{11} - 235238901940224a^{10}c^{11}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{12876603457536a^{11}c^{11} + 43797970944a^{12}c^{11} + 1058340146221982392995043868672c^{12}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-176775637161072795796969095168ac^{12} + 6235849644708280985967394816a^2c^{12}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{179112003462820022860382208a^3c^{12} - 7549631558688412740485120a^4c^{12}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-121866570019913213214720a^5c^{12} + 2500009663439970992128a^6c^{12}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{44613295101771841536a^7c^{12} - 113644325380325376a^8c^{12} - 4446228579287040a^9c^{12}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-14131384909824a^{10}c^{12} + 44710428672a^{11}c^{12} + 152076288a^{12}c^{12}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{50169135355853067670825992192c^{13} - 6709363518959579397365956608ac^{13}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{150614391320542327439622144a^2c^{13} + 7039163305933896564080640a^3c^{13}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-140953864332077212631040a^4c^{13} - 3711753500792413224960a^5c^{13}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{23530276519912931328a^6c^{13} + 762927749215027200a^7c^{13} + 1587081243525120a^8c^{13}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-31366085345280a^9c^{13} - 128024838144a^{10}c^{13} + 1974250608891543017637281792c^{14}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-207912674105719908850466816ac^{14} + 2433971414854765171965952a^2c^{14}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{197419671000565497200640a^3c^{14} - 1569222994723146301440a^4c^{14}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-73662625200783360000a^5c^{14} - 28046683124662272a^6c^{14} + 8006131050086400a^7c^{14}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{33986276229120a^8c^{14} - 93351444480a^9c^{14} - 381026304a^{10}c^{14}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{64395185753779071029870592c^{15} - 5229548355031289718374400ac^{15}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{14788626650875350220800a^2c^{15} + 4080138857445379276800a^3c^{15}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-1838738814365859840a^4c^{15} - 984309579582013440a^5c^{15} - 3963648988938240a^6c^{15}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{47795939573760a^7c^{15} + 243856834560a^8c^{15} + 1733754269178556886024192c^{16}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-105699998215764315209728ac^{16} - 462776974461593190400a^2c^{16}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{62047273223138574336a^3c^{16} + 274014892392579072a^4c^{16} - 8585594583121920a^5c^{16}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-51289950781440a^6c^{16} + 124468592640a^7c^{16} + 635043840a^8c^{16}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{38253074247649509507072c^{17} - 1690248262256299081728ac^{17}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-17030905158269140992a^2c^{17} + 677374598861291520a^3c^{17} + 5103823135703040a^4c^{17}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-44281532252160a^5c^{17} - 301234913280a^6c^{17} + 683995510182533660672c^{18}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-20883614573480902656ac^{18} - 300907292140765184a^2c^{18} + 5031796612792320a^3c^{18}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{47159301898240a^4c^{18} - 102503546880a^5c^{18} - 697303040a^6c^{18} + 9749520870090473472c^{19}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-192157335441899520ac^{19} - 3363268429086720a^2c^{19} + 22788156948480a^3c^{19}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{232532213760a^4c^{19} + 108101378949251072c^{20} - 1238931957874688ac^{20}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-24121180160000a^2c^{20} + 47475326976a^3c^{20} + 484442112a^4c^{20} + 898096988946432c^{21}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-4991691522048ac^{21} - 101871255552a^2c^{21} + 5257174188032c^{22} - 9453961216ac^{22}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-192937984a^2c^{22} + 19327352832c^{23} + 33554432c^{24}}{\Gamma(\frac{c-a}{2}) \Gamma(\frac{c+a+49}{2})} + \\
& + \frac{-174640699428026688529844729138380800000a}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{203146678984534054497568403939819520000a^2}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-85138755175820398653100504242008064000a^3}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{16061280659531143673149881316338278400a^4}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-1222084280207838423985096126241433600a^5}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-12283691196452377116468673849996800a^6 + 5784460407216745090330911562252800a^7}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-64163356321096906750752842736000a^8 - 13694126502362695192734160137600a^9}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{70187206573178654636297887200a^{10} + 19908972323990635614823432800a^{11}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{173269159811839104008127000a^{12} - 13877737266129829896585600a^{13}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-321996506691917154610800a^{14} + 1564939330380426184800a^{15}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{137080053389848041000a^{16} + 1531503700614518400a^{17} - 5877737170852800a^{18}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-269958340159200a^{19} - 2165643711000a^{20} - 2271561600a^{21} + 62053200a^{22}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{352800a^{23} + 600a^{24} + 174640699428027308978246462377820160000c}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-532864667571457665192973079183032320000ac}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{402516691346265375314108597722509312000a^2c}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-125889625791092176675310990146311475200a^3c}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{17789322238914085698760878421723806720a^4c}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-839642652359017269802420497774739200a^5c}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-43737799664111358026071194502939200a^6c + 4327054136964868650145069269816000a^7c}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{65710116059613413534021275073520a^8c - 8866517878966272728311508397200a^9c}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-150823836484592478601551835500a^{10}c + 8809059086367927385757986500a^{11}c}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{245730574725150062467057345a^{12}c - 2419735295216118363229200a^{13}c}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} +
\end{aligned}$$



$$\begin{aligned}
& + \frac{-158884196614329578449050a^{14}c - 1380872827191951285500a^{15}c}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{24710949541561727895a^{16}c + 584963403786946800a^{17}c + 3188363454621400a^{18}c}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-23871357856500a^{19}c - 392868365505a^{20}c - 1709061200a^{21}c - 409650a^{22}c}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{14700a^{23}c + 25a^{24}c + 329717988586925953482621073961779200000c^2}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-593610115863251160379838852807786496000ac^2}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{332339392507645324571342448478991155200a^2c^2}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-80292001276403892913036261386947788800a^3c^2}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{8435752253978765649343405490604441600a^4c^2}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-182800700903842751424899605150310400a^5c^2}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-27864925923991726587974486723020800a^6c^2}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{1153485575395825211251677531417600a^7c^2 + 54557531398454979304743054796800a^8c^2}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-1866238092614775212796148761600a^9c^2 - 82875721352461554443602089600a^{10}c^2}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{896036369449822368051211200a^{11}c^2 + 71047302433059217158129600a^{12}c^2}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{483566295038792118753600a^{13}c^2 - 21705016205569712558400a^{14}c^2}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-427069008643285008000a^{15}c^2 - 704037139627536000a^{16}c^2 + 54829120781635200a^{17}c^2}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{601725264940800a^{18}c^2 + 1380827448000a^{19}c^2 - 14486472000a^{20}c^2 - 100900800a^{21}c^2}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-187200a^{22}c^2 + 276232179692810109214204037425004544000c^3}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-361016488745751900835386008702130585600ac^3}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{157654746786562756857346716177587896320a^2c^3}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{-29811845754423340406338901058806169600a^3c^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+50}{2})} + \\
& + \frac{2261040469101812046662593980909465600a^4c^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+50}{2})} + \\
& + \frac{7467962566571130491482924613606400a^5c^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-8288810131500283801973700171383040a^6c^3 + 105753830113991285004211505699200a^7c^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+50}{2})} + \\
& + \frac{15276314790358723950186005280000a^8c^3 - 89839656535308780981746947200a^9c^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-16726883912057639605189273680a^{10}c^3 - 115201961703676996712982600a^{11}c^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+50}{2})} + \\
& + \frac{8257727829380832345907000a^{12}c^3 + 155470133092201748020200a^{13}c^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-603234005661019847400a^{14}c^3 - 41751662349615218000a^{15}c^3 - 358769923659518800a^{16}c^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+50}{2})} + \\
& + \frac{968095828299600a^{17}c^3 + 32206016242400a^{18}c^3 + 169856687000a^{19}c^3 + 106306200a^{20}c^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-1401400a^{21}c^3 - 2600a^{22}c^3 + 138505441369671545827613807716230758400c^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-141033120649691397441988498324822425600ac^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+50}{2})} + \\
& + \frac{49191424454401897055805697928999731200a^2c^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-7286784027262375583445490685229465600a^3c^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+50}{2})} + \\
& + \frac{374067588927556175588296387795353600a^4c^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+50}{2})} + \\
& + \frac{12066879415392989959435569486336000a^5c^4 - 1418367801582681846899974223616000a^6c^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-11724885340790270723926442035200a^7c^4 + 2225577070203302581076302156800a^8c^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+50}{2})} + \\
& + \frac{25892642522406895816692384000a^9c^4 - 1660403587242510892624752000a^{10}c^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-34718000105624284415366400a^{11}c^4 + 349058892421463336112000a^{12}c^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+50}{2})} + \\
& + \frac{15758070846864650304000a^{13}c^4 + 98029228370495136000a^{14}c^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+50}{2})} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{-1569321907433280000a^{15}c^4 - 25609443449990400a^{16}c^4 - 91184052960000a^{17}c^4}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{523867344000a^{18}c^4 + 4708704000a^{19}c^4 + 9609600a^{20}c^4}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{47103137180828963726258689199082307584c^5}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-38606916894738607152732025212449587200ac^5}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{10872909845856838905945144160621363200a^2c^5}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-1250646162787845304931184049334476800a^3c^5}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{37484137237073610387082179113287680a^4c^5}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{2960056649806039350518792618176000a^5c^5 - 147835927587805127993477304710400a^6c^5}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-4407777576011344380877953734400a^7c^5 + 181587707284941430929852497280a^8c^5}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{5386052499736584815450220000a^9c^5 - 73426510897677856161406800a^{10}c^5}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-3490081968585840976348000a^{11}c^5 - 12978357558398102190320a^{12}c^5}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{746347714949116280000a^{13}c^5 + 9937225737538642400a^{14}c^5 + 2454621732763200a^{15}c^5}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-738607265716320a^{16}c^5 - 4828187364000a^{17}c^5 - 4859654800a^{18}c^5 + 39239200a^{19}c^5}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{80080a^{20}c^5 + 11643512284338078214932291209134080000c^6}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-7811846265289713988464106586033356800ac^6}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{1782447335346527348520040105063219200a^2c^6}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-156185991237696882904399849104998400a^3c^6}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{1628516051137267451572910992588800a^4c^6 + 413130655084688778826418431795200a^5c^6}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-8682292288286157632177578598400a^6c^6 - 584337996155409991642835251200a^7c^6}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{6839225908971453044632166400a^8c^6 + 496205711149910054823129600a^9c^6}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{1041291387682609641177600a^{10}c^6 - 182911277098225554278400a^{11}c^6}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-2197804342078164019200a^{12}c^6 + 12956443561424563200a^{13}c^6}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{394556420664806400a^{14}c^6 + 1989973649664000a^{15}c^6 - 6690587904000a^{16}c^6}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-84756672000a^{17}c^6 - 192192000a^{18}c^6 + 2185434017754672893827554567467827200c^7}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-1210576372193996556349018510039449600ac^7}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{223356114720461583695372389167267840a^2c^7}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-14417840115170057077040994071756800a^3c^7 - 127654808190213133895037229670400a^4c^7}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{38362342001061342556663535539200a^5c^7 - 100943627312800952414696545280a^6c^7}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-45930295596168058484716070400a^7c^7 - 158519703458949712693632000a^8c^7}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{26491220954206657796347200a^9c^7 + 318516097490735070738240a^{10}c^7}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-4796976148767599596800a^{11}c^7 - 110735713090730041600a^{12}c^7}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-307118684131459200a^{13}c^7 + 7303674051273600a^{14}c^7 + 61439842464000a^{15}c^7}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{86692320000a^{16}c^7 - 504504000a^{17}c^7 - 1144000a^{18}c^7}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{320841683284681063430490974846976000c^8 - 147261685663349662978242993822105600ac^8}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{21828588787720013764469751624499200a^2c^8 - 978764458188092993647066703462400a^3c^8}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-29937782516869216684714136371200a^4c^8 + 2477913960753858299794587648000a^5c^8}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{30306732837796421151215616000a^6c^8 - 2324406168836210117372313600a^7c^8}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-35079004867678815190118400a^8c^8 + 810488727074359517184000a^9c^8}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{18840273819769239552000a^{10}c^8 - 11075412804186931200a^{11}c^8}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-2870202835480780800a^{12}c^8 - 20168019615744000a^{13}c^8 + 36128581632000a^{14}c^8}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{774918144000a^{15}c^8 + 1976832000a^{16}c^8 + 37623854094699788409092847834234880c^9}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-14310486246486860625946551595827200ac^9 + 1685406368191632966927765317222400a^2c^9}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-46774053309260554179902883430400a^3c^9 - 2907367898999056189584016384000a^4c^9}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{110290157599761586095108096000a^5c^9 + 3091438308324915618903142400a^6c^9}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-73000241928734665517772800a^7c^9 - 2137975103648864229235200a^8c^9}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{9101205323954551808000a^9c^9 + 599787058295388620800a^{10}c^9 + 3364607652652646400a^{11}c^9}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-36478880044006400a^{12}c^9 - 421246938112000a^{13}c^9 - 788646144000a^{14}c^9}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{3587584000a^{15}c^9 + 9152000a^{16}c^9 + 3578017643742734189270453452800000c^{10}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-1124822208152756932629830094028800ac^{10} + 103529588791248473877009294950400a^2c^{10}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-1311215108521671070577904844800a^3c^{10} - 185621293383786411465336422400a^4c^{10}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{3060077972270587621992038400a^5c^{10} + 168536740537058850462105600a^6c^{10}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-1016343899030933407334400a^7c^{10} - 75771265876540543795200a^8c^{10}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-312238666563644620800a^9c^{10} + 10967326380122112000a^{10}c^{10}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{110687000154931200a^{11}c^{10} - 58633012838400a^{12}c^{10} - 4098455961600a^{13}c^{10}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-11948851200a^{14}c^{10} + 278986102350768311768576111411200c^{11}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-72124888712454702873593459507200ac^{11} + 5065553589978581619224540610560a^2c^{11}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{5329959949354438689384038400a^3c^{11} - 8535575626577387683612262400a^4c^{11}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{25100649148683283563315200a^5c^{11} + 6059556534266084019322880a^6c^{11}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{22155491991978909491200a^7c^{11} - 1690530802659461734400a^8c^{11}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-16415314891981824000a^9c^{11} + 95043876381941760a^{10}c^{11} + 1706929285734400a^{11}c^{11}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{4157702348800a^{12}c^{11} - 15524454400a^{13}c^{11} - 45260800a^{14}c^{11}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{17973019590141218234189217792000c^{12} - 3792934708034192465188198809600ac^{12}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{196285879020447610271642419200a^2c^{12} + 2786214843661751921698406400a^3c^{12}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-289801563760060600614912000a^4c^{12} - 2064705306471178076160000a^5c^{12}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{148846508105105620992000a^6c^{12} + 1585317292830248140800a^7c^{12}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-22299454819830988800a^8c^{12} - 354397212868608000a^9c^{12} - 241573183488000a^{10}c^{12}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{13413128601600a^{11}c^{12} + 45622886400a^{12}c^{12} + 961450433202482165846785392640c^{13}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-163998227183988503762121523200ac^{13} + 5924648410475117616444211200a^2c^{13}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{166515545915840350896128000a^3c^{13} - 7213269573432725603942400a^4c^{13}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-115037909882834792448000a^5c^{13} + 2404631605453956300800a^6c^{13}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{42663988630042214400a^7c^{13} - 110264137559654400a^8c^{13} - 4272497037312000a^9c^{13}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-13576756838400a^{10}c^{13} + 42990796800a^{11}c^{13} + 146227200a^{12}c^{13}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{42819108327436116421509120000c^{14} - 5828749054688624947023052800ac^{14}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{133998221660012686186905600a^2c^{14} + 6158133813035578687488000a^3c^{14}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-125789151018504683520000a^4c^{14} - 3283349863865057280000a^5c^{14}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{21132740770489958400a^6c^{14} + 679691867258880000a^7c^{14} + 1409416298496000a^8c^{14}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{-28005433344000a^9c^{14} - 114307891200a^{10}c^{14} + 1588225843907503003480883200c^{15}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-169762020930124137680076800ac^{15} + 2043378481535587823124480a^2c^{15}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{162454135634000379904000a^3c^{15} - 1316763562211776921600a^4c^{15}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-61099456493769523200a^5c^{15} - 21663400701788160a^6c^{15} + 6667626921984000a^7c^{15}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{28300728729600a^8c^{15} - 77792870400a^9c^{15} - 317521920a^{10}c^{15}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{48967663640865914486784000c^{16} - 4025992923028302436761600ac^{16}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{12152046118550057779200a^2c^{16} + 3163650334517152972800a^3c^{16}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-1616946504361574400a^4c^{16} - 767423554781184000a^5c^{16} - 3085932036096000a^6c^{16}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{37340577792000a^7c^{16} + 190513152000a^8c^{16} + 1249348776871366450216960c^{17}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-76939510335577862963200ac^{17} - 329033249485284966400a^2c^{17}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{45434381571765043200a^3c^{17} + 199713870087782400a^4c^{17} - 6309093310464000a^5c^{17}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-37687050240000a^6c^{17} + 91521024000a^7c^{17} + 466944000a^8c^{17}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{26180456293639127040000c^{18} - 1166186969396202700800ac^{18}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-11694085683491635200a^2c^{18} + 469496995774464000a^3c^{18} + 3535117221888000a^4c^{18}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-30751064064000a^5c^{18} - 209190912000a^6c^{18} + 445504624400741171200c^{19}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-13687619181241958400ac^{19} - 196960707910041600a^2c^{19} + 3308414369792000a^3c^{19}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{31005671424000a^4c^{19} - 67436544000a^5c^{19} - 458752000a^6c^{19} + 6054208717455360000c^{20}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-119879948147097600ac^{20} - 2097585900748800a^2c^{20} + 14242598092800a^3c^{20}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{145332633600a^4c^{20} + 64105204122910720c^{21} - 737026192179200ac^{21}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{-14349002342400a^2c^{21} + 28259123200a^3c^{21} + 288358400a^4c^{21} + 509356277760000c^{22}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-2836188364800ac^{22} - 57881395200a^2c^{22} + 2855482163200c^{23} - 5138022400ac^{23}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} + \\
& + \frac{-104857600a^2c^{23} + 10066329600c^{24} + 16777216c^{25}}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+50}{2})} \Big]. \quad (8)
\end{aligned}$$

### §3. Derivation of summation formulae

Derivation of main result (8):

Substituting  $b = -a - 49, z = \frac{1}{2}$  in given result (2), we get

$$\begin{aligned}
& (2a + 49) {}_2F_1 \left[ \begin{matrix} a, & -a - 49 & ; & \frac{1}{2} \\ c & & & \end{matrix} \right] \\
& = a {}_2F_1 \left[ \begin{matrix} a + 1, & -a - 49 & ; & \frac{1}{2} \\ c & & & \end{matrix} \right] + (a + 49) {}_2F_1 \left[ \begin{matrix} a, & -a - 48 & ; & \frac{1}{2} \\ c & & & \end{matrix} \right].
\end{aligned}$$

Now using same parallel method which is used in [6], we can prove the main result.

### References

- [1] A. P. Prudnikov, Y. A. Brychkov and O. I. Marichev, More special functions, Nauka, Moscow, 1986, translated from the Russian by G. G. Gould, Gordon and Breach Science Publishers, New York, Philadelphia, London, Paris, Montreux, Tokyo, Melbourne, Integrals and Series, **3**(1990).
- [2] Asish Arora, Rahul Singh and Salahuddin, Development of a family of summation formulae of half argument using Gauss and Bailey theorems, Journal of Rajasthan Academy of Physical Sciences, **7**(2008), 335-342.
- [3] E. D. Rainville, The contiguous function relations for  ${}_pF_q$  with applications to Bateman's  $J_n^{u,v}$  and Rice's  $H_n(\zeta, p, \nu)$ , Bull. Amer. Math. Soc., **51**(1945), 714-723.
- [4] L. C. Andrews, Special function of mathematics for engineers, second edition, McGraw-Hill Co Inc., New York, 1992.
- [5] Richard Bells and Roderick Wong, Special functions, a graduate text, Cambridge Studies in Advanced Mathematics, 2010.
- [6] Salahuddin and M. P. Chaudhary, A new summation formula allied with hypergeometric function, Global Journal of Science Frontier Research, **11**(2010), 21-37.
- [7] Salahuddin, Evaluation of a summation formula involving recurrence relation, Gen. Math. Notes., **2**(2010), 42-59.



# Coefficient bounds for certain subclasses

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**Abstract** In this paper, we introduce some subclasses of analytic functions and determine the sharp upper bounds of the functional  $|a_2a_4 - a_3^2|$  for the functions of the form  $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$  belonging to these classes in the unit disc  $E = \{z : |z| < 1\}$ .

**Keywords** Analytic functions, starlike functions, convex functions, alpha-convex functions, functions whose derivative has a positive real part, Bazilevic functions, Hankel determinant.

## §1. Introduction and preliminaries

Let  $A$  be the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1)$$

in the unit disc  $E = \{z : |z| < 1\}$ .

Let  $S$  be the class of functions  $f(z) \in A$  and univalent in  $E$ .

Let  $M_{\alpha}(\alpha \geq 0)$  be the class of functions in  $A$  which satisfy the conditions

$$\frac{f(z)f'(z)}{z} \neq 0$$

and

$$Re \left[ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \frac{(zf'(z))'}{f'(z)} \right] > 0. \quad (2)$$

The class  $M_{\alpha}$  was introduced by Mocanu<sup>[15]</sup> and functions of this class are called  $\alpha$ -convex functions. Obviously  $M_0 \equiv S^*$ , the class of starlike functions and  $M_1 \equiv K$ , the class of convex functions. Miller, Mocanu and Reade<sup>[14]</sup> have shown that  $\alpha$ -convex functions are starlike in  $E$ , and for  $\alpha \geq 1$ , all  $\alpha$ -convex functions are convex in  $E$ . Therefore  $\alpha$ -convex functions are also called  $\alpha$ -starlike functions. Concept of  $\alpha$ -convex functions gives a continuous parametrization between starlike functions and convex functions.

$H_{\alpha}(\alpha \geq 0)$  be the class of functions in  $A$  which satisfy the condition

$$Re \left[ (1 - \alpha)f'(z) + \alpha \frac{(zf'(z))'}{f'(z)} \right] > 0. \quad (3)$$

This class was introduced by Al-Amiri and Reade <sup>[1]</sup>. In particular  $H_0 \equiv R$ , the class of functions whose derivative has a positive real part and studied by Macgregor <sup>[12]</sup>. Also  $H_1 \equiv K$ .

$B_\alpha (\alpha \geq 0)$  is the class of functions in  $A$  which satisfy the condition

$$Re \left[ f'(z) \left( \frac{f(z)}{z} \right)^{\alpha-1} \right] > 0. \quad (4)$$

The class  $B_\alpha$  was introduced by Singh <sup>[19]</sup> and studied further by Thomas <sup>[21]</sup> and El-Ashwah and Thomas <sup>[3]</sup>. Functions of this class are called Bazilevic functions. Particularly  $B_0 \equiv S^*$  and  $B_1 \equiv R$ .

In 1976, Noonan and Thomas <sup>[16]</sup> stated the  $q$ th Hankel determinant for  $q \geq 1$  and  $n \geq 1$  as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n+q-1} & \dots & \dots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has also been considered by several authors. For example, Noor <sup>[17]</sup> determined the rate of growth of  $H_q(n)$  as  $n \rightarrow \infty$  for functions given by Eq.(1) with bounded boundary. Ehrenborg <sup>[2]</sup> studied the Hankel determinant of exponential polynomials and the Hankel transform of an integer sequence is defined and some of its properties discussed by Layman <sup>[9]</sup>. Also Hankel determinant for different classes was studied by various authors including Hayman <sup>[5]</sup>, Pommerenke <sup>[18]</sup>, Janteng et al.<sup>[6,7,8]</sup> and recently by Mehrok and Singh <sup>[13]</sup>.

Easily, one can observe that the Fekete-Szegő functional is  $H_2(1)$ . Fekete and Szegő <sup>[4]</sup> then further generalised the estimate of  $|a_3 - \mu a_2^2|$  where  $\mu$  is real and  $f \in S$ . For our discussion in this paper, we consider the Hankel determinant in the case of  $q = 2$  and  $n = 2$ ,

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

For  $\mu$  complex, Szynal <sup>[21]</sup> obtained the estimates for  $|a_3 - \mu a_2^2|$  for the class  $M_\alpha$ . Al-Amiri and Reade <sup>[1]</sup> obtained the estimates for  $|a_3 - \mu a_2^2|$  for the class  $H_\alpha$  and Singh <sup>[19]</sup> obtained the estimates for  $|a_3 - \mu a_2^2|$  for the class  $B_\alpha$ .

In this paper, we seek upper bound of the functional  $|a_2 a_4 - a_3^2|$  for the functions belonging to the classes  $M_\alpha$ ,  $H_\alpha$  and  $B_\alpha$ . Results due to various authors follow as special cases.

## §2. Main result

Let  $P$  be the family of all functions  $p$  analytic in  $E$  for which  $Re(p(z)) > 0 (z \in E)$  and

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots \quad (5)$$

**Lemma 2.1.** If  $p \in P$ , then  $|p_k| \leq 2 (k = 1, 2, 3, \dots)$ .

This result is due to Pommerenke <sup>[18]</sup>.

**Lemma 2.2.** If  $p \in P$ , then

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for some  $x$  and  $z$  satisfying  $|x| \leq 1$ ,  $|z| \leq 1$  and  $p_1 \in [0, 2]$ .

This result was proved by Libera and Zlotkiewicz <sup>[10,11]</sup>.

**Theorem 2.1.** If  $f \in M_\alpha$ , then

$$|a_2a_4 - a_3^2| \leq \frac{1}{(1+2\alpha)^2} \left[ \frac{3\alpha(1+\alpha)^3}{(1+3\alpha)(2+15\alpha+24\alpha^2+7\alpha^3)} + 1 \right]. \quad (6)$$

**Proof.** As  $f \in M_\alpha$ , so from (2)

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} = p(z). \quad (7)$$

On expanding and equating the coefficients of  $z$ ,  $z^2$  and  $z^3$  in (7), we obtain

$$a_2 = \frac{p_1}{1+\alpha}, \quad (8)$$

$$a_3 = \frac{p_2}{2(1+2\alpha)} + \frac{(1+3\alpha)p_1^2}{2(1+2\alpha)(1+\alpha)^2} \quad (9)$$

and

$$a_4 = \frac{p_3}{3(1+3\alpha)} + \frac{(1+5\alpha)p_1p_2}{2(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{(4+6\alpha+17\alpha^2)p_1^3}{6(1+2\alpha)(1+3\alpha)(1+\alpha)^3}. \quad (10)$$

Using (8), (9) and (10), it yields

$$|a_2a_4 - a_3^2| = \frac{1}{C(\alpha)} \left| \frac{4(1+2\alpha)^2(1+\alpha)^3p_1(4p_3) + 12(1+2\alpha)(1+5\alpha)(1+\alpha)^2p_1^2(2p_2)}{+8(1+2\alpha)(1+6\alpha+17\alpha^2)^2p_1^4 - 3(1+3\alpha)((1+\alpha)^2(2p_2) + 2(1+3\alpha)p_1^2)^2} \right|. \quad (11)$$

where  $C(\alpha) = 48(1+3\alpha)(1+2\alpha)^2(1+\alpha)^4$ .

Using lemma 2.1 and lemma 2.2 in (11) and replacing  $p_1$  by  $p$ , it can be easily established that

$$|a_2a_4 - a_3^2| \leq \frac{1}{C(\alpha)} \left[ \begin{aligned} &[-4(1+2\alpha)(1+\alpha)^2 - 12(1+2\alpha)(1+5\alpha)(1+\alpha)^2 - 8(1+2\alpha)(1+6\alpha+17\alpha^2) \\ &\quad + 3(1+3\alpha)(3+8\alpha+\alpha^2)]p^4 + [8(1+2\alpha)^2(1+\alpha)^3 + 12(1+2\alpha)(1+5\alpha) \\ &\quad - 6(1+3\alpha)(3+8\alpha+\alpha^2)(1+\alpha)^2]p^2(4-p^2)\delta + (1+\alpha)^3(2-p)[6(1+\alpha)(1+3\alpha) \\ &\quad - (1+4\alpha+7\alpha^2)](4-p^2)p^2\delta^2 + 8(1+2\alpha)^2(1+\alpha)^3p(4-p^2) \end{aligned} \right],$$

where  $\delta = |x| \leq 1$ .

Therefore

$$|a_2a_4 - a_3^2| \leq \frac{1}{C(\alpha)} F(\delta),$$

where

$$F(\delta) = [-4(1+2\alpha)(1+\alpha)^2 - 12(1+2\alpha)(1+5\alpha)(1+\alpha)^2 - 8(1+2\alpha)(1+6\alpha+17\alpha^2) + 3(1+3\alpha)(3+8\alpha+\alpha^2)]p^4 + [8(1+2\alpha)^2(1+\alpha)^3 + 12(1+2\alpha)(1+5\alpha) - 6(1+3\alpha)(3+8\alpha+\alpha^2)(1+\alpha)^2]p^2(4-p^2)\delta + (1+\alpha)^3(2-p)[6(1+\alpha)(1+3\alpha) - (1+4\alpha+7\alpha^2)](4-p^2)p^2\delta^2 + 8(1+2\alpha)^2(1+\alpha)^3p(4-p^2).$$

As  $F'(\delta) > 0$ , so  $F(\delta)$  is an increasing function in  $[0, 1]$  and therefore  $\text{Max}F(\delta) = F(1)$ .

Consequently

$$|a_2a_4 - a_3^2| \leq \frac{1}{C(\alpha)}G(p), \quad (12)$$

where  $G(p) = F(1)$ .

So

$$G(p) = -A(\alpha)p^4 + B(\alpha)p^2 + 48(1+3\alpha)(1+\alpha)^4,$$

where

$$A(\alpha) = 4\alpha(1+\alpha)(2+15\alpha+24\alpha^2+7\alpha^3) \text{ and } B(\alpha) = 48\alpha(1+\alpha)^4.$$

$$\text{Now } G'(p) = -4A(\alpha)p^3 + 2B(\alpha)p \text{ and } G''(p) = -12A(\alpha)p^2 + 2B(\alpha).$$

$$G'(p) = 0 \text{ gives } p[2A(\alpha)p^2 - B(\alpha)] = 0.$$

$$G''(p) \text{ is negative at } p = \sqrt{\frac{6(1+\alpha)^3}{(2+15\alpha+24\alpha^2+7\alpha^3)}} = p'.$$

$$\text{So } \text{Max}G(p) = G(p').$$

Hence from (12), we obtain (6).

The result is sharp for  $p_1 = p'$ ,  $p_2 = p_1^2 - 2$  and  $p_3 = p_1(2 - p_1^2)$ .

For  $\alpha = 0$  and  $\alpha = 1$  respectively, theorem 2.1 gives the following results due to Janteng et al.<sup>[8]</sup>.

**Corollary 2.1.** If  $f(z) \in S^*$ , then

$$|a_2a_4 - a_3^2| \leq 1.$$

**Corollary 2.2.** If  $f(z) \in K$ , then

$$|a_2a_4 - a_3^2| \leq \frac{1}{8}.$$

**Theorem 2.2.** If  $f \in H_\alpha$ , then

$$|a_2a_4 - a_3^2| \leq \begin{cases} \frac{4}{9(1+\alpha)^2} & \text{if } 0 \leq \alpha \leq \frac{5}{17}, \\ \frac{(17\alpha-5)^2}{144(1+2\alpha)(1+20\alpha+7\alpha^2-4\alpha^3)} + \frac{4}{9(1+\alpha)^2} & \text{if } \frac{5}{17} \leq \alpha \leq 1. \end{cases} \quad (13)$$

**Proof.** Since  $f \in H_\alpha$ , so from (3)

$$(1-\alpha)f'(z) + \alpha \frac{(zf'(z))'}{f'(z)} = p(z). \quad (14)$$

On expanding and equating the coefficients of  $z$ ,  $z^2$  and  $z^3$  in (14), we obtain

$$a_2 = \frac{p_1}{2}, \quad (15)$$

$$a_3 = \frac{p_2 + \alpha p_1^2}{3(1+\alpha)} \quad (16)$$

and

$$a_4 = \frac{p_3}{4(1+2\alpha)} + \frac{3\alpha p_1 p_2}{4(1+\alpha)(1+2\alpha)} + \frac{\alpha(2\alpha-1)p_1^3}{4(1+\alpha)(1+2\alpha)}. \quad (17)$$

Using (15), (16) and (17), it yields

$$|a_2 a_4 - a_3^2| = \frac{1}{C(\alpha)} \left| \begin{aligned} &9(1+\alpha)^3 p_1(4p_3) + 54(1+\alpha)p_1^2(2p_2) \\ &+ 36\alpha(2\alpha-1)(1+\alpha)p_1^4 - 8(1+2\alpha)(2p_2+2\alpha p_1^2)^2 \end{aligned} \right|, \quad (18)$$

where  $C(\alpha) = 288(1+3\alpha)(1+2\alpha)(1+\alpha)^2$ .

Using lemma 2.1 and lemma 2.2 in (18) and replacing  $p_1$  by  $p$ , we obtain

$$|a_2 a_4 - a_3^2| \leq \frac{1}{C(\alpha)} \left[ \begin{aligned} &(1-12\alpha+3\alpha^2+8\alpha^3)p^4 + 2(1+13\alpha+4\alpha^2)p^2(4-p^2)\delta \\ &+ (2-p)[16(1+2\alpha) - (1+2\alpha+9\alpha^2)p^2](4-p^2)\delta^2 + 18(1+\alpha)^2 p(4-p^2) \end{aligned} \right],$$

where  $\delta = |x| \leq 1$ .

Therefore

$$|a_2 a_4 - a_3^2| \leq \frac{1}{C(\alpha)} F(\delta),$$

where

$$F(\delta) = (1-12\alpha+3\alpha^2+8\alpha^3)p^4 + 2(1+13\alpha+4\alpha^2)p^2(4-p^2)\delta + (2-p)[16(1+2\alpha) - (1+2\alpha+9\alpha^2)p^2](4-p^2)\delta^2 + 18(1+\alpha)^2 p(4-p^2)$$

is an increasing function.

Therefore  $\text{Max}F(\delta) = F(1)$ . Consequently

$$|a_2 a_4 - a_3^2| \leq \frac{1}{C(\alpha)} G(p), \quad (19)$$

where  $G(p) = F(1)$ .

So

$$G(p) = -A(\alpha)p^4 + B(\alpha)p^2 + 128(1+2\alpha),$$

where  $A(\alpha) = 2(1+20\alpha+17\alpha^2-4\alpha^3)$  and  $B(\alpha) = 4(1+\alpha)(17\alpha-5)$ .

**Case I.** For  $0 \leq \alpha \leq \frac{5}{17}$ ,  $B(\alpha) < 0$ .

So  $G(p)$  is maximum at  $p = 0$  and it follows the first result of (13).

In this case, the result is sharp for  $p_1 = 0$ ,  $p_2 = -2$  and  $p_3 = 0$ .

**Case II.** For  $\frac{5}{17} \leq \alpha \leq 1$ , as in theorem 2.1,  $G(p)$  is maximum for  $p = \sqrt{\frac{(1+\alpha)(17\alpha-5)}{(1+20\alpha+7\alpha^2-4\alpha^3)}} = p'$ , so  $\text{Max}G(p) = G(p')$ .

In this case, the result is sharp for  $p_1 = p'$ ,  $p_2 = p_1^2 - 2$  and  $p_3 = p_1(p_1^2 - 3)$ .

Hence the theorem. For  $\alpha = 0$  in theorem 2.2, we obtain the following results due to Janteng et al.<sup>[6]</sup>.

**Corollary 2.3.** If  $f(z) \in R$ , then

$$|a_2 a_4 - a_3^2| \leq \frac{4}{9}.$$

Putting  $\alpha = 1$  in theorem 2.2, we get the following results due to Janteng et al.<sup>[8]</sup>.

**Corollary 2.4.** If  $f(z) \in K$ , then

$$|a_2a_4 - a_3^2| \leq \frac{1}{8}.$$

On the same lines, we can easily prove the following theorem:

**Theorem 2.3.** If  $f \in B_\alpha$ , then

$$|a_2a_4 - a_3^2| \leq \frac{4}{(2 + \alpha)^2}.$$

The result is sharp for  $p_1 = 0$ ,  $p_2 = -2$  and  $p_3 = 0$ .

For  $\alpha = 0$ , theorem 2.3 gives the following result due to Janteng et al.<sup>[8]</sup>.

**Corollary 2.5.** If  $f(z) \in S^*$ , then

$$|a_2a_4 - a_3^2| \leq 1.$$

For  $\alpha = 0$ , theorem 2.3 gives the following result due to Janteng et al.<sup>[6]</sup>.

**Corollary 2.6.** If  $f(z) \in R$ , then

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}.$$

## References

- [1] H. S. Al-Amiri and M. O. Reade, On a linear combination of some expressions in the theory of the univalent functions, *Monatshafte fur Mathematik*, **80**(1975), 257-264.
- [2] R. Ehrenborg, The Hankel determinant of exponential polynomials, *American Mathematical Monthly*, **107**(2000), 557-560.
- [3] R. M. El-Ashwah and D. K. Thomas, Growth results for a subclass of Bzilevic functions, *Int. J. Math. and Math. Sci.*, **8**(1985), 785-793.
- [4] M. Fekete and G. Szegő, Eine Bemerkung über ungerade schlichte Funktionen, *J. London Math. Soc.*, **8**(1933), 85-89.
- [5] W. K. Hayman, *Multivalent functions*, Cambridge Tracts in Math. and Math. Phys., Cambridge University Press, No.48, 1958.
- [6] Aini Janteng, Suzeini Abdul Halim and Maslina Darus, Coefficient inequality for a function whose derivative has a positive real part, *J. Ineq. in Pure and Appl. Math.*, **7**(2006), No. 2, 1-5.
- [7] Aini Janteng, Suzeini Abdul Halim and Maslina Darus, Hankel determinant for functions starlike and convex with respect to symmetric points, *J. Quality Measurement and Anal.*, **2**(2006), No. 1, 37-43.
- [8] Aini Janteng, Suzeini Abdul Halim and Maslina Darus, Hankel determinant for starlike and convex functions, *Int. J. Math. Anal.*, **1**(2007), No. 13, 619-625.
- [9] J. W. Layman, The Hankel transform and some of its properties, *J. of integer sequences*, **4**(2001), 1-11.
- [10] R. J. Libera and E. J. Zlotkiewicz, Early coefficients of the inverse of a regular convex function, *Proc. Amer. Math. Soc.*, **85**(1982), 225-230.

- [11] R. J. Libera and E. J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative in  $P$ , Proc. Amer. Math. Soc., **87**(1983), 251-257.
- [12] T. H. MacGregor, Functions whose derivative has a positive real part, Tras. Amer. Math. Soc., **104**(1962), 532-537.
- [13] B. S. Mehrok and Gagandeep Singh, Estimate of second Hankel determinant for certain classes of analytic functions, Scientia Magna, **8**(2012), No. 3, 85-94.
- [14] S. S. Miller, P. T. Mocanu and M. O. Reade, All  $\alpha$ -convex functions are univalent and starlike, Proc. Amer. Math. Soc., **37**(1973), 553-554.
- [15] P. T. Mocanu, "Une propriete' de convexite' g n ralis e dans la th orie de la repr sentation conforme", Mathematica(CLUJ), **11**(1969), No. 34, 127-133.
- [16] J. W. Noonan and D. K. Thomas, On the second Hankel determinant of a really mean  $p$ -valent functions, Trans. Amer. Math. Soc., **223**(1976), No. 2, 337-346.
- [17] K. I. Noor, Hankel determinant problem for the class of functions with bounded boundary rotation, Rev. Roum. Math. Pures Et Appl., **28**(1983), No. 8, 731-739.
- [18] C. Pommerenke, Univalent functions, G ttingen: Vandenhoeck and Ruprecht., 1975.
- [19] R. Singh, On Bazilevic functions, Proc. Amer. Math. Soc., **38**(1973), 261-273.
- [20] J. Szynal, Some remarks on coefficient inequality for  $\alpha$ -convex functions, Bull. De L'Acad. Des Sci Ser des Sci Math. Astr et phys., **20**(1972), No. 11, 917-919.
- [21] D. K. Thomas, On a subclass of Bazilevic functions, Int. J. Math. and Math. Sci., **8**(1985), 779-783.

# Certain regression models involving univariate maximal and minimal survival functions on mixed censored survival data

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**Abstract** Mixed censored survival data arise from taking the maximum or minimum of two-component or even multi-component systems. Tied to this is the need to estimate maximal or minimal survival functions from related marginal survival functions. In this study we exploit the fact that the transformation of survival time dynamics of one component to another can be considered to result from variations in shift (mean), concentration (mode), spread (variance), degree of symmetry (skewness) and peakedness (kurtosis) as well as the underlying relative proportion of the censored data. Consequently, we propose some regression models that can serve for prediction purposes. The models are evaluated using a combination of some real data from the literature combined with some Weibull simulated data of varying levels of censoring and parameters. To the extent that the performance measures are excellent, the suggested models can be effective for prediction analysis.

**Keywords** Maximal survival function, minimal survival function, mixed censored survival data, univariate.

**2000 Mathematics Subject Classification:** 62N01, 62N02.

## §1. Introduction and preliminaries

Mixed censored survival data arise from taking the maximum or minimum of two-component systems or even multi-components. Tied to this is the need to estimate maximal or minimal survival functions from related marginal survival functions. Bivariate survival data for two-component systems pervade many fields of scientific research. For example, medical researchers have obtained data on the infection times of the left and right kidneys of patients, and times to HIV infection of sex partners. Clayton (1978) applied regression model for censored survival data to epidemiological studies of chronic disease incidence. A related model for association in bivariate survivorship time distributions was proposed for the analysis of familial tendency in disease incidence.<sup>[3]</sup> The possible extension of the model to general multivariate survivorship distribution was indicated. Oakes (1982) discussed a reparameterization of a model introduced by Clayton in bivariate life-tables and the inference for the parameter governing the association



was considered when the marginal distributions are specified up to Lehmman alternatives.<sup>[5]</sup>

The information matrix was derived explicitly and it was shown that the parameterization is moderately successful in introducing orthogonality between the associated parameters. Demographers have collected data on the two survival times of individual twins. Economists interested in modeling the joint retirement decisions of married couples have collected data of times to retirement for both the wife and the husband of each surveyed household. One of the common features shared by taking the maximum or minimum of these data is a mixture of the two duration times, calling for a joint distribution that allows a probability mass on the  $45^\circ$  line in the sample space. The exploitation of facts of the transformation of survival time dynamics of one component to another which will result to variations in shift (mean), concentration (mode), spread (variance), degree of symmetry (skewness) and peakedness (kurtosis) as well as the underlying relative proportion of the censored data is the purpose of this study. Specifically, we propose some regression models that can serve for prediction purpose.

In [4], the likelihood function of the estimation from censored survival data was given and several examples are seen. The type of estimate studied here can be briefly indicated as follows. Suppose from a two-component system a random sample of  $N$  paired subjects are taken such that  $T_{11}, T_{12}, \dots, T_{1N}$  and  $T_{21}, T_{22}, \dots, T_{2N}$  are two random variables. Taking the maximum and minimum between the observed lifetimes separately will result to two sets of mixed samples. The sample distribution functions  $\hat{F}_{Y_2}(t)$  and  $\hat{F}_{Y_1}(t)$  are naturally defined as that which assigns a probability of  $\frac{1}{N}$  to each of the given values, so that  $\hat{F}_{Y_2}(t)$  equals  $\frac{1}{N}$  times the number of sample values less than the argument  $t_{Y_2}$  and  $\hat{F}_{Y_1}(t)$  equals  $\frac{1}{N}$  times the number of sample values less than the argument  $t_{Y_1}$ . Besides describing the sample, they are also non-parametric estimates of the mixed population distributions. When observations are incomplete the corresponding estimates are still step functions with discontinuities at the ages of observed deaths, but can no longer be obtained as a mere description of the sample. The two samples are incomplete in the sense that some items in the mixtures have given not random samples  $T_{11}, T_{12}, \dots, T_{1N}$  and  $T_{21}, T_{22}, \dots, T_{2N}$  of random variables  $T_1$  and  $T_2$  itself (called the lifetimes), but the observed lifetimes  $Z_i = (Z_{1i}, Z_{2i})$  where  $Z_{1i} = \min(T_{1i}, C_{1i})$  and  $Z_{2i} = \min(T_{2i}, C_{2i})$ ;  $i = 1, 2, \dots, N$ . Here  $C_{1i}$  and  $C_{2i}$ , called limits of observation (i.e. censored times), are constants or values of the other random variables, which are assumed to be independent of  $T_{1i}$  and  $T_{2i}$  respectively (unless otherwise stated). For each item it is known whether one has

$$T_{1i} \leq C_{1i}, Z_{1i} = T_{1i} \quad \text{and} \quad T_{2i} \leq C_{2i}, Z_{2i} = T_{2i}, \quad (\text{deaths}),$$

or

$$T_{1i} > C_{1i}, Z_{1i} = C_{1i} \quad \text{and} \quad T_{2i} > C_{2i}, Z_{2i} = C_{2i}, \quad (\text{losses}).$$

Ordinarily the  $T_{1i}, T_{2i}, C_{1i}$  and  $C_{2i}$  are so defined as to be necessarily non-negative. The mixed censored survival data is such that an item in the mixture belongs to one population with certain probability and to the other with another probability. This results into a division of mutually exclusive classes of deaths and losses. A loss by definition always precludes the desired knowledge of  $\vec{T}_i = (T_{1i}, T_{2i})$ . On the other hand, a death does not preclude the knowledge of the corresponding  $\vec{C}_i = (C_{1i}, C_{2i})$ .

**Notation 1.1.** Let  $T_{1i}$  and  $C_{1i}$  represent the failure and censoring time for the  $i$ th subject of the first component. Also let  $T_{2i}$  and  $C_{2i}$  represent the failure and censoring time for the  $i$ th subject of the second component. Let the bivariate vector failure time  $\vec{T}_i$  have cumulative distribution function  $F_i = (F_{1i}, F_{2i})$ , probability density function  $f_i = (f_{1i}, f_{2i})$  and survivor function  $\bar{F}_i = (\bar{F}_{1i}, \bar{F}_{2i})$ . Also let the bivariate censoring life time  $\vec{C}_i$  have cumulative distribution function, cumulative distribution function  $G_i = (G_{1i}, G_{2i})$ , probability density function  $g_i = (g_{1i}, g_{2i})$  and survivor function  $\bar{G}_i = (\bar{G}_{1i}, \bar{G}_{2i})$ . Let  $Y_i$  be the corresponding vector of covariates. Let  $T_i = (T_{1i}, T_{2i})$  and  $C_i = (C_{1i}, C_{2i})$  be independently conditional on  $Y_i$ . On each of  $N$  individuals, we observe the group  $(Y_i, \delta_{1i}, \delta_{2i}, \gamma_i)$  where  $Z_i = (Z_{1i}, Z_{2i}) =$  denotes the follow up time for the paired entity on individual  $i$  under study.

$\delta_{1i}$  = denotes a censoring indicator for the first member of an entity on individual  $i$ .

$\delta_{2i}$  = denotes a censoring indicator for the second member of an entity on individual  $i$ .

$\gamma_i$  = the binary indicator used to indicate which of the following up time is considered on individual  $i$ .

That is,

$$\gamma_i = \begin{cases} 1, & \text{if } Z_{1i} \leq Z_{2i}, \\ 0, & \text{if } Z_{1i} > Z_{2i}. \end{cases}$$

$L_{\max}(Y, \delta_1, \delta_2, \gamma)$  = the likelihood function for the maximal survival function and  $L_{\min}(Y, \delta_1, \delta_2, \gamma)$  = the likelihood function for the minimal survival function.

Observing the maximum between the follow up time for the paired entity on the  $i$ th individual (i.e.  $Y_i = \max(Z_{1i}, Z_{2i})$ ), we have the likelihood function, for the  $N$  i.i.d groups  $(Y, \delta_1, \delta_2, \gamma)$ , for the maximum survival function as

$$\begin{aligned} L_{\max}(Y, \delta_1, \delta_2, \gamma) &= \prod_{i=1}^n \{ (g_{1i}(y) \bar{F}_{1i}(y))^{[(1-\delta_{1i})(1-\gamma_i)]} \times (g_{2i}(y) \bar{F}_{2i}(y))^{[(1-\delta_{2i})\gamma_i]} \\ &\quad \times (f_{2i}(y) \bar{G}_{2i}(y))^{[\delta_{2i}\gamma_i]} \times (f_{1i}(y) \bar{G}_{1i}(y))^{[\delta_{1i}(1-\gamma_i)]} \\ &\quad \times \left( \int_0^y g_{2i}(\tau) \bar{F}_{2i}(\tau) d\tau \right)^{[(1-\delta_{2i})(1-\gamma_i)]} \times \left( \int_0^y g_{1i}(\tau) \bar{F}_{1i}(\tau) d\tau \right)^{[(1-\delta_{1i})\gamma_i]} \\ &\quad \times \left( \int_0^y f_{2i}(\tau) \bar{G}_{2i}(\tau) d\tau \right)^{[\delta_{2i}(1-\gamma_i)]} \times \left( \int_0^y f_{1i}(\tau) \bar{G}_{1i}(\tau) d\tau \right)^{[\delta_{1i}\gamma_i]} \}. \quad (1) \end{aligned}$$

In the same light, observation of the minimum follow up time for the paired entity also on the  $i$ th individual (i.e.  $Y_i = \min(Z_{1i}, Z_{2i})$ ) results in the likelihood function for the minimal survival function given by

$$\begin{aligned} L_{\min}(Y, \delta_1, \delta_2, \gamma) &= \prod_{i=1}^n \{ (g_{1i}(y) \bar{F}_{1i}(y))^{[(1-\delta_{1i})\gamma_i]} \times (g_{2i}(y) \bar{F}_{2i}(y))^{[(1-\delta_{2i})(1-\gamma_i)]} \\ &\quad \times (f_{2i}(y) \bar{G}_{2i}(y))^{[\delta_{2i}(1-\gamma_i)]} \times (f_{1i}(y) \bar{G}_{1i}(y))^{[\delta_{1i}\gamma_i]} \\ &\quad \times \left( \int_y^\infty g_{2i}(\tau) \bar{F}_{2i}(\tau) d\tau \right)^{[(1-\delta_{2i})\gamma_i]} \times \left( \int_y^\infty g_{1i}(\tau) \bar{F}_{1i}(\tau) d\tau \right)^{[(1-\delta_{1i})(1-\gamma_i)]} \\ &\quad \times \left( \int_y^\infty f_{2i}(\tau) \bar{G}_{2i}(\tau) d\tau \right)^{[\delta_{2i}\gamma_i]} \times \left( \int_y^\infty f_{1i}(\tau) \bar{G}_{1i}(\tau) d\tau \right)^{[\delta_{1i}(1-\gamma_i)]} \}. \quad (2) \end{aligned}$$

Defined for various censored and failure situations.

**Notation 1.2.**

$$\begin{aligned} l_{i0} &= g_i(y) \bar{F}_i(y) = Pr\{C_i = y, T_i > y\}; \quad i = 1, 2, \\ l_{i1} &= f_i(y) \bar{G}_i(y) = Pr\{T_i = y, C_i > y\}; \quad i = 1, 2. \\ L_{ij}(y) &= \int_0^y l_{ij}(\tau) d\tau \quad \text{and} \quad \bar{L}_{ij}(y) = \int_y^\infty l_{ij}(\tau) d\tau; \quad i = 1, 2, \quad j = 0, 1. \end{aligned}$$

It is clear that the terms of  $L_{\max}(Y, \delta_1, \delta_2, \gamma)$  and  $L_{\min}(Y, \delta_1, \delta_2, \gamma)$  can be expressed as

$$\begin{aligned} L_{\max}(Y, \delta_1, \delta_2, \gamma) &= \prod_{i=1}^n \{ (l_{10}(y))^{[(1-\delta_{1i})(1-\gamma_i)]} \times (l_{20}(y))^{[(1-\delta_{2i})\gamma_i]} \\ &\quad \times (l_{21}(y))^{[\delta_{2i}\gamma_i]} \times (l_{11}(y))^{[\delta_{1i}(1-\gamma_i)]} \times (L_{10}(y))^{[(1-\delta_{1i})\gamma_i]} \\ &\quad \times (L_{20}(y))^{[(1-\delta_{2i})(1-\gamma_i)]} \times (L_{21}(y))^{[\delta_{2i}(1-\gamma_i)]} \times (L_{11}(y))^{[\delta_{1i}\gamma_i]} \}, \quad (3) \end{aligned}$$

$$\begin{aligned} L_{\min}(Y, \delta_1, \delta_2, \gamma) &= \prod_{i=1}^n \{ (l_{10}(y))^{[(1-\delta_{1i})\gamma_i]} \times (l_{20}(y))^{[(1-\delta_{2i})(1-\gamma_i)]} \\ &\quad \times (l_{21}(y))^{[\delta_{2i}(1-\gamma_i)]} \times (l_{11}(y))^{[\delta_{1i}(1-\gamma_i)]} \times (\bar{L}_{10}(y))^{[(1-\delta_{1i})\gamma_i]} \\ &\quad \times (\bar{L}_{20}(y))^{[(1-\delta_{2i})(1-\gamma_i)]} \times (\bar{L}_{21}(y))^{[\delta_{2i}(1-\gamma_i)]} \times (\bar{L}_{11}(y))^{[\delta_{1i}\gamma_i]} \}. \quad (4) \end{aligned}$$

Dividing (3) by (4), we have (5) as

$$\begin{aligned} \frac{L_{\max}(Y, \delta_1, \delta_2, \gamma)}{L_{\min}(Y, \delta_1, \delta_2, \gamma)} &= \left\{ \left( \frac{l_{11}(y)}{L_{11}(y)} \right)^{[\delta_{1i}(1-\gamma)]} \times \left( \frac{L_{20}(y)}{l_{20}(y)} \right)^{[(1-\delta_{2i})(1-\gamma)]} \right. \\ &\quad \times \left( \frac{l_{20}(y)}{\bar{L}_{20}(y)} \right)^{[(1-\delta_{2i})\gamma]} \times \left( \frac{L_{11}(y)}{l_{11}(y)} \right)^{[\delta_{1i}\gamma]} \times \left( \frac{l_{10}(y)}{\bar{L}_{10}(y)} \right)^{[(1-\delta_{1i})(1-\gamma)]} \\ &\quad \left. \times \left( \frac{L_{21}(y)}{l_{21}(y)} \right)^{[\delta_{2i}(1-\gamma)]} \times \left( \frac{l_{21}(y)}{\bar{L}_{21}(y)} \right)^{[\delta_{2i}\gamma]} \times \left( \frac{L_{10}(y)}{l_{10}(y)} \right)^{[(1-\delta_{2i})\gamma]} \right\}. \quad (5) \end{aligned}$$

We introduce the following notations

$$\begin{aligned} m_{ij} &= \frac{l_{ij}(y)}{\bar{L}_{ij}(y)}; \quad i = 1, 2; \quad j = 0, 1, \\ n_{ij} &= \frac{l_{ij}(y)}{L_{ij}(y)}; \quad i = 1, 2; \quad j = 0, 1. \end{aligned}$$

**Remark 1.1.** In light of this notations, likelihood functions here in this section, the ratio of the likelihood function of the the maximal and minimal survival functions will be given in Sections 2. It is established that any regression of maximal survival functions on minimal survival functions is essentially a relation of certain instantaneous censored and failure rates. The validation of these models (which will be seen in Section 2) was such that for fixed first component data of proportion  $p = \frac{58}{70}$  of censoring to Weibull distribution  $(\beta, \eta)$ , the second component data was generated for varied levels of censoring of proportion  $p = \frac{58}{70}, \frac{35}{70}$ , and  $\frac{12}{70}$  with mean parameters  $\lambda$  and  $.75\lambda$ . We thus generally have 6 combinations identified as  $(a, b, c, d) \equiv$  (mean of real first component data, proportion of censored points in first component data, mean of simulated second component data, proportion of censored

points in the second component data). The marginal, maximal and minimal survival function for the two components was obtained and appropriate regression analysis was carried out. The results will be presented also in Section 2. Discussions and conclusions will be made. The study concludes that the eight models developed in this study gave good hypothesis and can reliably predict minimal survival function and maximal survival function from marginal survival functions and recommends that models such as the ones discussed herewith, in particular Models 2.7, 2.4 and 2.8 (the best performing models) be adopted for studying failure times in phenomena that can be modelled in terms of components.

## §2. Main results

Using these notations in (5), we have the ratio of the likelihood of the maximal and minimal survival functions given by

$$\frac{L_{\max}(Y, \delta_1, \delta_2, \gamma)}{L_{\min}(Y, \delta_1, \delta_2, \gamma)} = \left( \frac{(m_{10}(y))^{(1-\delta_1)}(m_{11}(y))^{\delta_1}}{(n_{20}(y))^{(1-\delta_2)}(n_{21}(y))^{\delta_2}} \right)^{1-\gamma} \left( \frac{(m_{20}(y))^{(1-\delta_2)}(m_{21}(y))^{\delta_2}}{(n_{10}(y))^{(1-\delta_1)}(n_{11}(y))^{\delta_1}} \right)^{\gamma}. \quad (6)$$

(6) breaks down, for specific values of  $\delta_1, \delta_2, \gamma$  into the following ratio for likelihoods of maximum and minimal survival functions e.g.

$$\frac{L_{\max}(y|\delta_1 = 0, \delta_2 = 0, \gamma = 0)}{L_{\min}(y|\delta_1 = 0, \delta_2 = 0, \gamma = 0)} = \frac{m_{10}(y)}{n_{20}(y)}.$$

It is noted that  $m_{ij}(y)$  and  $n_{ij}(y)$ ,  $i = 1, 2$ ;  $j = 0, 1$  can be likened to instantaneous event rates. (6) shows that the ratio of the likelihoods of the maximal and minimal survival function can be expressed as ratio of instantaneous censored and failure rates. Any regression of maximal survival functions on minimal survival functions is essentially a relation of certain instantaneous censored and failure rates.

For the purpose of exploiting this relation, it is necessary we state some results that suggest appropriate models of interest.

**Theorem 2.1.** Let  $X_1, X_2$  be stochastically independent random variables with distribution function  $F_1(\cdot)$  and  $F_2(\cdot)$  respectively. Also, let maximum  $Y_2 = \max(X_1, X_2)$  and the minimum  $Y_1 = \min(X_1, X_2)$  be ordered statistics respectively having the distribution functions  $G_2(\cdot)$  and  $G_1(\cdot)$  respectively. Then for  $Y_2$  and  $Y_1$  we have that  $G_2(t) = F_1(t)F_2(t)$  and  $G_1(t) = 1 - ((1 - F_1(t))(1 - F_2(t)))$ .

**Proof.**  $G_2(t) = Pr(Y_2 < t) = Pr(X_1 < t; X_2 < t) = Pr(X_1 < t)Pr(X_2 < t)$  (by stochastic independence)

$$= F_1(t)F_2(t). \quad (7)$$

Similarly

$$G_1(t) = 1 - Pr(Y_1 > t) = 1 - Pr(X_1 > t, X_2 > t) = 1 - (1 - F_1(t))(1 - F_2(t)). \quad (8)$$

**Model 2.1.** The (8) suggests the regression model  $\ln \bar{F}_{Y_2}(t) = \alpha_1 \ln \bar{F}_{1i}(t) + \alpha_2 \ln \bar{F}_{2i}(t) + \ln \varepsilon_i$ , where  $\varepsilon_i \sim N(0, 1)$  and  $\ln = \log_e$ ,  $\bar{F}_{Y_1}(t) = Pr(Y_1 > t) = \bar{F}_{\min(X_1, X_2)}(t)$ ,  $\bar{F}_1(t) = Pr(X_1 > t)$  and  $\bar{F}_2(t) = Pr(X_2 > t)$ .

**Model 2.2.** The (7) suggests the regression model  $\ln \bar{F}_{Y_{1i}}(t) = \beta_1 \ln \bar{F}_{1i}(t) + \beta_2 \ln \bar{F}_{2i}(t) + \ln \tau_i$ , where  $\tau_i \sim N(0, 1)$  and  $\ln = \log_e$ ,  $\bar{F}_{Y_2}(t) = Pr(Y_2 > t) = \bar{F}_{\max(X_1, X_2)}(t)$ ,  $\bar{F}_1(t) = Pr(X_1 > t)$  and  $\bar{F}_2(t) = Pr(X_2 > t)$ .

**Theorem 2.2.** If  $\bar{F}_x(t)$  is a survivor function, then  $\bar{F}_Y(t) = \frac{\beta \bar{F}_x(t)}{1 - (1 - \beta) \bar{F}_x(t)}$ ,  $t > 0$  is a proper survivor function for all  $0 < \beta \leq 1$ .

**Proof.** This follows trivially from the definition.

From this relation, we have that

$$\frac{\beta(1 - \bar{F}_Y(t))}{\bar{F}_Y(t)} = \frac{1 - \bar{F}_X(t)}{\bar{F}_X(t)}. \quad (9)$$

(9) can be assumed to hold for

1.  $\bar{F}_Y(t) \equiv \bar{F}_{\max(X_1, X_2)}(t)$  against  $\bar{F}_X(t) \equiv \bar{F}_{X_1}(t)$ ,
2.  $\bar{F}_Y(t) \equiv \bar{F}_{\max(X_1, X_2)}(t)$  against  $\bar{F}_X(t) \equiv \bar{F}_{X_2}(t)$ ,
3.  $\bar{F}_Y(t) \equiv \bar{F}_{\min(X_1, X_2)}(t)$  against  $\bar{F}_X(t) \equiv \bar{F}_{X_1}(t)$ ,
4.  $\bar{F}_Y(t) \equiv \bar{F}_{\min(X_1, X_2)}(t)$  against  $\bar{F}_X(t) \equiv \bar{F}_{X_2}(t)$ .

In view of all these we formulate the following set of regression models.

**Model 2.3.**

$$\frac{1 - \bar{F}_{Y_{2i}}(t)}{\bar{F}_{Y_{2i}}(t)} = \gamma_1 \frac{(1 - \bar{F}_{1i}(t))}{\bar{F}_{1i}(t)} + \gamma_2 \frac{(1 - \bar{F}_{2i}(t))}{\bar{F}_{2i}(t)} + \tau_i,$$

where  $\tau_i \sim N(0, 1)$  and  $\bar{F}_{Y_2}(t) \equiv \bar{F}_{\max(X_1, X_2)}(t)$ .

**Model 2.4.**

$$\frac{1 - \bar{F}_{Y_{1i}}(t)}{\bar{F}_{Y_{1i}}(t)} = \alpha_1 \frac{(1 - \bar{F}_{1i}(t))}{\bar{F}_{1i}(t)} + \alpha_2 \frac{(1 - \bar{F}_{Y_{2i}}(t))}{\bar{F}_{2i}(t)} + \varepsilon_i,$$

where  $\varepsilon_i \sim N(0, 1)$  and  $\bar{F}_{Y_1}(t) \equiv \bar{F}_{\min(X_1, X_2)}(t)$ .

**Theorem 2.3.** Assume that the random variable  $X$  is a symmetric triangular distribution with the probability density function symmetric about  $h$  given by

$$h(x) = \begin{cases} \frac{a+(x-h)}{a^2}, & h-a < x < h, \\ \frac{a-(x-h)}{a^2}, & h < x < h+a, \end{cases}$$

then the corresponding cumulative distribution is given by

$$H(x) = \begin{cases} \frac{1}{2a^2}(x - (h-a))^2, & h-a < x < h, \\ \frac{1}{2a^2}(a^2 + 2a(x-h) - (x-h)^2), & h < x < h+a. \end{cases}$$

**Proof.** See [1].

**Theorem 2.4.** Let  $f_0(x)$  be a unimodal probability density function,  $H(x)$  a cumulative distribution function having probability density function  $h(x)$ , and assume both  $f_0(x)$  and  $h(x)$  are symmetric about zero.

Let  $f(x) = \frac{2(1+\alpha x^2)}{1+\alpha k} f_0(x) H(w(x))$ ,  $-\infty < x < \infty$  and  $k = \int_{-\infty}^{\infty} x^2 f_0(x) dx$ , then  $f(x)$  is a probability density function for any odd function  $w(x)$  and any  $\alpha > 0$ .

**Proof.** See [6].

**Corollary 2.1.** In the particular case that  $f_0(x)$  in Theorem 2.4 is a probability density function  $w(x) = x$ ,  $h(x) = \frac{1}{2}(1 - \frac{|x|}{a}) I_{-a,a}(x)$ , so that

$$H(x) = \begin{cases} \frac{1}{2a^2}(a^2 + 2ax + x^2), & -a < x < 0, \\ \frac{1}{2a^2}(a^2 + 2ax - x^2), & 0 < x < a. \end{cases}$$

The function

$$f(x) = \begin{cases} \frac{1+\alpha x^2}{\alpha^2(1+\alpha k)} f_0(x)(a^2 + 2ax + x^2), & -a < x < 0, \\ \frac{1+\alpha x^2}{\alpha^2(1+\alpha k)} f_0(x)(a^2 + 2ax - x^2), & 0 < x < a, \end{cases}$$

is also a probability density function. <sup>[1]</sup>

**Remark 2.1.** We note as follows.

1. the relationship in Corollary 2.1 can be simply expressed as  $f(x) = \sum_{i=1}^2 a_i x^i f_0(x)$  for certain co-efficient  $a_i$ ,  $i = 1, 2$ .
2. The practical effect of Corollary 2.1 is to generate a possibly asymmetric and/or bimodal density distribution from a symmetric and unimodal density.
3. A survival function may essentially be geometrically different from another survival function by such characteristics (measures) as shift (mean), concentration (mode), spread (variance), asymmetry (skewness) and peakedness (kurtosis) all of which can be captured in the mechanism of the transformation in Models 2.5 and 2.6.

**Model 2.5.**

$$\bar{F}_{Y_{2i}}(t) = \sum_{i=1}^2 a_i t^i \bar{F}_{1i}(t) + \sum_{i=1}^2 b_i t^i \bar{F}_{2i}(t) + \sum_{i=1}^2 m_i t^i \bar{F}_{1i}(t) \bar{F}_{2i}(t) + \varepsilon_i, \text{ where } \varepsilon_i \sim N(0, 1).$$

**Model 2.6.**

$$\bar{F}_{Y_{1i}}(t) = \sum_{i=1}^2 c_i t^i \bar{F}_{1i}(t) + \sum_{i=1}^2 d_i t^i \bar{F}_{2i}(t) + \sum_{i=1}^2 n_i t^i \bar{F}_{1i}(t) \bar{F}_{2i}(t) + \tau_i, \text{ where } \tau_i \sim N(0, 1).$$

**Theorem 2.5.** If  $\bar{F}(t)$  is a survival function, so also is, if  $\bar{F}^*(t)$  defined by

$$\bar{F}^*(t) = e^{[-(\frac{1}{\bar{F}(t)})^{\theta-1} + 1]}, \theta > 1.$$

**Proof.** This follows from the definition of survival function. In view of Theorem 2.5 we postulate that  $\bar{F}_{\max}(t)$  can be expressed in terms of  $\bar{F}_1(t)$  and  $\bar{F}_2(t)$  as follows

$$\bar{F}_{\max}(t) = \left( e^{[-(\frac{1}{\bar{F}(t)})^{(\theta_1-1)} + 1]} \times e^{[-(\frac{1}{\bar{F}(t)})^{(\theta_2-1)} + 1]} \right). \quad (10)$$

Taking logarithm of (10), we have

$$\ln \bar{F}_{\max}(t) = 2 - (\bar{F}_1(t))^{1-\theta_1} - (\bar{F}_2(t))^{1-\theta_2},$$

where  $\theta_1, \theta_2 > 1$  and  $\ln = \log_e$ .

We can thus develop a regression model as such

**Model 2.7.**

$$\ln \bar{F}_{Y_{2i}}(t) = \beta_0 + \alpha_1(\bar{F}_{1i}(t))^{-\frac{1}{4}} + \alpha_2(\bar{F}_{1i}(t))^{-\frac{3}{4}} + \alpha_3(\bar{F}_{2i}(t))^{-\frac{1}{4}} + \alpha_4(\bar{F}_{2i}(t))^{-\frac{3}{4}} + \varepsilon_i, \text{ where } \varepsilon_i \sim N(0, 1).$$

**Model 2.8.**

$$\ln \bar{F}_{Y_{1i}}(t) = \beta_0 + \gamma_1(\bar{F}_{1i}(t))^{-\frac{1}{4}} + \gamma_2(\bar{F}_{1i}(t))^{-\frac{3}{4}} + \gamma_3(\bar{F}_{2i}(t))^{-\frac{1}{4}} + \gamma_4(\bar{F}_{2i}(t))^{-\frac{3}{4}} + \tau_i, \text{ where } \tau_i \sim N(0, 1).$$

These sets of models were validated using a real data for first marginal component and simulated data for second marginal component. The parameters are generally somewhat correlated, it may thus not be advisable to independently choose the parameters in a modelling situation. For example, the maximum likelihood estimate for the real data assuming Weibull [7] distribution for time to failure is given as  $\hat{\beta} = 1.823823334$  and  $\hat{\eta} = 5666.376110$ . In the light of these properties a set of data with some shape parameter ( $\beta$ ) but double the scale parameter ( $\eta$ ) will have double the mean. This point proved useful in the simulation exercise. The maximum likelihood estimates of the Weibull distribution for the real data were obtained and thus its mean ( $\lambda$ , say). Considered to be of special interest is the extent to which these sets of models were validated to different mix (proportion) of censored data and for different closeness or balance of the marginal components. The proportion  $p$  of censored data were varied for different amounts  $p = 58/70, 35/70$  and  $12/70$ . In the same way different combinations of the marginal components were considered. Thus, we had cases  $(\lambda, \lambda)$  and  $(\lambda, 75\lambda)$ . This is to say that the distribution of the first marginal component is equal with that of the second marginal component in terms of the mean of the Weibull distribution and where there is a shift in the sense that the second marginal component is only .75 of the mean. This elucidated situations when censored points for the second component was high ( $p = 58/70$ ) just like the original data; censored points were even ( $p = 35/70$ ) and censored points were sparse ( $p = 12/70$ ). In summary, each of the regression models were validated with six generated samples of the second marginal component against the real data from the literature adopted for the first marginal component. Among the statistical tools that were employed as appropriate for the validation of the performances of these models were the adjusted  $R^2$  (adj.  $R^2$ ) and error sum of squares (SSE). The necessary results are given on Table 2.1 for various levels of censoring i.e., highly, evenly and sparsely censored as well as for equal and shift in mean. Models 2.1, 2.3, 2.5 and 2.7 connect maximal survival functions to the marginal survival functions while Models 2.2, 2.4, 2.6 and 2.8 connect minimal survival functions to marginal survival functions.

**Table 2.1.** Adjusted  $R^2$  and Residual Sum of Squares Values for the Maximal and Minimal Response function by level of censoring of Simulated and Shift of Mean for the Models relative to the real data (from combination  $(\lambda, 58/70; \lambda, 58/70)$  to combination  $(\lambda, 58/70; .75\lambda, 12/70)$ ).

| Intensity (Mean) | Response Variable | Highly-censored |      | Evenly-censored |          | Sparsely-censored |      |
|------------------|-------------------|-----------------|------|-----------------|----------|-------------------|------|
|                  |                   | Adj. $R^2$      | SSE  | Adj. $R^2$      | SSE      | Adj. $R^2$        | SSE  |
| Equal Mean       | Model 2.1.        | .980            | .003 | .979            | .018     | .879              | .034 |
|                  | Model 2.3.        | .977            | .004 | .964            | .054     | .968              | .191 |
|                  | Model 2.5.        | .909            | .446 | .889            | .446     | .896              | .378 |
|                  | Model 2.7.        | .992            | .000 | .997            | .001     | 1.000             | .000 |
|                  | Model 2.2.        | .985            | .003 | .951            | .018     | .998              | .010 |
|                  | Model 2.4.        | .985            | .004 | .965            | .029     | .977              | .034 |
|                  | Model 2.6.        | .904            | .470 | .901            | .468     | .912              | .391 |
|                  | Model 2.8.        | .996            | .000 | .994            | .001     | .999              | .000 |
| Shifted Mean     | Model 2.1.        | .989            | .001 | .937            | .008     | .968              | .014 |
|                  | Model 2.3.        | .992            | .001 | .979            | .011     | .971              | .019 |
|                  | Model 2.5.        | .929            | .370 | .964            | .160     | .948              | .184 |
|                  | Model 2.7.        | .993            | .000 | .963            | .004     | .935              | .009 |
|                  | Model 2.2.        | .998            | .001 | .805            | 27.814   | .993              | .083 |
|                  | Model 2.4.        | .999            | .001 | .761            | 1.844    | .999              | .333 |
|                  | Model 2.6.        | .916            | .355 | .951            | 1776.725 | .920              | .213 |
|                  | Model 2.8.        | .999            | .000 | .998            | .003     | .989              | .071 |

The study have been able to achieve the formulation of regression models based on the original thought that every survival function is a transformation of another survival function by such characteristics (measure) as shift (mean), concentration (mode), spread (variance), degree of symmetry (skewness) and peakedness (kurtosis). Based on the analysis of the data used, the following conclusions were reached.

- (1) all the sets of regression models can reliably predict minimal survival functions and maximal survival functions from marginal survival functions.
- (2) irrespective of the level of censoring and shift in mean, Models 2.4. and 2.8. perform excellently in prediction of minimal response functions and the best overall is Model 2.8 with adjusted  $R^2$  in the range (.989 - .999). Model 2.8 compared with Model 2.6 in the range (.901 - .951). Similarly, in terms of residual sum of squares Model 2.8 is in the range (.000 - .071) and Model 2.6 is in the range (.213 - 1776.725) irrespective of whether it is equal or shifted mean.

However for the prediction of maximal response function, the performance of Model 2.7 in relation to the others is less pronounced for shifted mean compared with when we have equal mean. This is especially so when using the criterion of the adjusted  $R^2$ . It is noted that Models 2.5 and 2.6 which makes allowance for bimodality have not performed well in relation to the others but it performed best where the proportion of censoring when the two components are closely equally censored (i.e. in the case where the data is approaching bimodal).



It is recommended that models such as the ones informed by (9) and (10), in particular 2.7, 2.4 and 2.8 (the best performing models) be adopted for studying failure times in phenomena that can be modelled in terms of components. This is particularly so in the transformation when the survival time dynamics of one component to another component can be considered to be shift (mean), concentration (mode), spread (variance), degree of symmetry (skewness) and peakedness (kurtosis).

## References

- [1] S. O. Adioha, Development of regression models involving univariate maximal and minimal survival functions on mixed censored survival data, An M.Sc. thesis in the department of mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria, 2010. .
- [2] P. Bulla, P. Muliere and S. Walker, Bayesian nonparametric estimation of a bivariate survival function, *Journal of Statistica Sinica*, **17**(2007), 427-444.
- [3] D. G. Clayton, A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence, *Biometrika*, **65**(1978), 141-151.
- [4] E. L. Kaplan and P. Meier, Nonparametric estimation from incomplete observations, *Journal of the American Statistical Association*, **53**(1958), 457-481.
- [5] D. Oakes, A model for association in bivariate survival data, *Journal of Statistical Society*, **44**(1982), 414-422.
- [6] A. A. Olosunde, The families of distributions arising from some skewing mechanisms, An M.Sc. thesis in the department of mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria, 2010.
- [7] N. Wayne, *Applied life data analysis*, Newyork, John Wiley and Sons, Inc, 1982.
- [8] J. H. Zar, *Biostatistical analysis*, 4th ed. India Pearson Education Inc, 1999.

# Fekete-Szegő Inequality for a Subclass of $p$ -valently Alpha Convex Functions

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**Abstract** In the present paper, we obtain sharp upper bound of the functional  $|a_{p+2} - \mu a_{p+1}^2|$  ( $\mu$  real) for a subclass of  $p$ -valently alpha convex functions and deduce several results in the form of corollaries and remarks.

**Keywords** Subordination, bounded functions,  $p$ -valent and univalent functions, starlike functions, convex functions and  $\alpha$ -convex functions.

**2000 Mathematics Subject Classification:** 30C45, 30C50.

## §1. Introduction

By  $U$ , we denote the class of bounded analytic functions  $w(z)$  in the unit disc  $E = \{z : |z| < 1\}$  and of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, \quad z \in E, \quad (1)$$

which satisfying the conditions  $w(0) = 0, |w(z)| < 1$ .

It is well known that

$$|c_1| \leq 1, |c_2| \leq 1 - |c_1|^2. \quad (2)$$

Let  $f(z)$  and  $F(z)$  be two analytic functions in the unit disc  $E$ , then  $f(z)$  is said to be subordinate to  $F(z)$  if there exists a function  $w(z) \in U$  such that  $f(z) = F(w(z))$  and we write as  $f(z) \prec F(z)$ .

In case the function  $F$  is univalent, the above subordination is equivalent to  $f(0) = F(0)$  and  $f(E) \subset F(E)$ .

Let  $A_p(p$  is a positive integer) denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \quad (3)$$

which are analytic in the unit disc  $E$ .

By  $A$ , we denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (4)$$

analytic in the unit disc  $E$ .

$S_p^*$  represents the class of  $p$ -valently starlike functions  $f(z)$  in  $A_p$  and satisfying the condition

$$Re \left[ \frac{zf'(z)}{f(z)} \right] > 0, \quad z \in E. \quad (5)$$

$S_p^*(A, B)$  represents the class of functions  $f(z)$  in  $A_p$  which satisfy the condition

$$\frac{zf'(z)}{f(z)} \prec p \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad z \in E. \quad (6)$$

Clearly  $S_p^*(1, -1) \equiv S_p^*$ , which shows that  $S_p^*(A, B)$  is a subclass of  $S_p^*$ . To avoid repetition, we lay down, once for all that  $-1 \leq B < A \leq 1$  and  $z \in E$ .

$S^*$  is the class of analytic univalent starlike functions  $f(z)$  in  $A$  such that

$$Re \left[ \frac{zf'(z)}{f(z)} \right] > 0. \quad (7)$$

$S^*(A, B)$  is the subclass of univalent starlike functions  $f(z)$  in  $A$  such that

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}. \quad (8)$$

Janowski [6] introduced the class  $S^*(A, B)$  and proved some results by the method of variational techniques. Goel and Mehrook [2,3] studied the class  $S^*(A, B)$  using subordination principle.

The class  $K_p$  consists of functions  $f(z)$  in  $A_p$  for which

$$Re \left[ \frac{(zf'(z))'}{f'(z)} \right] > 0. \quad (9)$$

Clearly  $f(z) \in K_p$  implies that  $\frac{zf'(z)}{p} \in S_p^*$ .

$K_p(A, B)$  is the subclass of functions  $f(z)$  in  $A_p$  for which

$$\frac{(zf'(z))'}{f'(z)} \prec p \frac{1 + Az}{1 + Bz}. \quad (10)$$

$K$  stands for the class of convex univalent functions  $f(z)$  in  $A$  with the condition

$$Re \left[ \frac{(zf'(z))'}{f'(z)} \right] > 0. \quad (11)$$

$K(A, B)$  is the subclass of  $K$  of functions  $f(z)$  in  $A$  with the condition

$$\frac{(zf'(z))'}{f'(z)} \prec \frac{1 + Az}{1 + Bz}. \quad (12)$$

Let  $M_p(\alpha)$  ( $\alpha \geq 0$ ) denote the class of functions  $f(z)$  in  $A_p$  which satisfy the conditions  $\frac{f(z) \cdot f'(z)}{z^{2p-1}} \neq 0$  and

$$Re \left[ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \frac{(zf'(z))'}{f'(z)} \right] > 0. \quad (13)$$

$M_p(\alpha; A, B)$  ( $\alpha \geq 0$ ) denotes the subclass of  $M_p(\alpha)$  of functions  $f(z)$  in  $A_p$  and satisfying the conditions  $\frac{f(z) \cdot f'(z)}{z^{2p-1}} \neq 0$  and

$$(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \frac{(zf'(z))'}{f'(z)} \prec p \frac{1 + Az}{1 + Bz}. \quad (14)$$

$M(\alpha)$  denotes the class of functions  $f(z)$  in  $A$  and satisfying the conditions  $\frac{f(z) \cdot f'(z)}{z} \neq 0$  and

$$\operatorname{Re} \left[ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \frac{(zf'(z))'}{f'(z)} \right] > 0. \quad (15)$$

The class  $M(\alpha)$  was introduced by Mocanu<sup>[9]</sup>. Miller et al.<sup>[8]</sup> have shown that all  $\alpha$ -convex functions are starlike in  $E$  and for  $\alpha \geq 1$ , all  $\alpha$ -convex functions are convex in  $E$ .

$M(\alpha; A, B)$  ( $\alpha \geq 0$ ) is the subclass of  $M(\alpha)$  of functions  $f(z)$  in  $A$  and satisfying the conditions  $\frac{f(z) \cdot f'(z)}{z} \neq 0$  and

$$(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \frac{(zf'(z))'}{f'(z)} \prec p \frac{1 + Az}{1 + Bz}. \quad (16)$$

The class  $M(\alpha; A, B)$  was studied by Goel and Mehrotra<sup>[4,5]</sup>. Fekete and Szegő<sup>[1]</sup> made an early study for the bounds of  $|a_3 - \mu a_2^2|$  ( $\mu$  real) when  $f(z)$  is analytic univalent in  $E$ . The well known result due to them states that if  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  is analytic and univalent in  $E$ , then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), & 0 \leq \mu \leq 1; \\ 4\mu - 3, & \mu \geq 1. \end{cases}$$

The following observations are obvious:

- (i)  $M_p(\alpha; 1, -1) \equiv M_p(\alpha)$ .
- (ii)  $M_1(\alpha; A, B) \equiv M(\alpha; A, B)$ .
- (iii)  $M_p(0; A, B) \equiv S_p^*(A, B)$ .
- (iv)  $M_p(0; 1, -1) \equiv S_p^*$ .
- (v)  $M_1(0; A, B) \equiv S^*(A, B)$ .
- (vi)  $M_1(0; 1, -1) \equiv S^*$ .
- (vii)  $M_p(1; A, B) \equiv K_p(A, B)$ .
- (viii)  $M_p(1; 1, -1) \equiv K_p$ .
- (ix)  $M_1(1; A, B) \equiv K(A, B)$ .
- (x)  $M_1(1; 1, -1) \equiv K$ .

## §2. Main results

**Theorem 2.1.** Let  $f(z) \in M_p(\alpha; A, B)$ , then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} \frac{(A-B)^2 p^4}{(p+\alpha)^2} (\lambda - \mu), & \mu \leq \lambda - \nu; \\ \frac{(A-B)p^2}{2(p+2\alpha)}, & \lambda - \nu \leq \mu \leq \lambda + \nu; \\ \frac{(A-B)^2 p^4}{(p+\alpha)^2} (\mu - \lambda), & \mu \geq \lambda + \nu. \end{cases} \quad (17)$$

where

$$\lambda = \frac{p(A-B)(p^2 + 2p\alpha + \alpha) - B(p+\alpha)^2}{2p^2(A-B)(p+2\alpha)} \quad (18)$$

and

$$\nu = \frac{(p+\alpha)^2}{2p^2(A-B)(p+2\alpha)}. \quad (19)$$

Results are sharp.

**Proof.** By definition of  $M_p(\alpha; A, B)$ , we have

$$(1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \frac{(zf'(z))'}{f'(z)} = p \frac{1 + Aw(z)}{1 + Bw(z)}.$$

Expanding the series, we get

$$\begin{aligned} & (1-\alpha)[p + a_{p+1}z + (2a_{p+2} - a_{p+1}^2)z^2 + \dots] + \alpha \left[ p + \frac{(p+1)}{p}a_{p+1}z \right] \\ & + \alpha \left[ \frac{2(p+2)}{p}a_{p+2} - \frac{(p+1)^2}{p^2}a_{p+1}^2 \right] z^2 + \dots \\ & = p[1 + (A-B)c_1z + (A-B)(c_2 - Bc_1^2)z^2 + \dots]. \end{aligned} \quad (20)$$

Equating the coefficients of  $z$  and  $z^2$  in (20), we obtain

$$a_{p+1} = \frac{(A-B)p^2c_1}{(p+\alpha)} \quad (21)$$

and

$$a_{p+2} = \frac{(A-B)p^2c_2}{2(p+2\alpha)} + \frac{(A-B)^2p^3(p^2 + 2p\alpha + \alpha)c_1^2}{2(p+2\alpha)(p+\alpha)^2} - \frac{Bp(A-B)c_1^2}{2(p+2\alpha)}. \quad (22)$$

From equations (21) and (22), we obtain

$$a_{p+2} - \mu a_{p+1}^2 = \frac{(A-B)p^2c_2}{2(p+2\alpha)} + \frac{(A-B)^2p^4}{(p+\alpha)^2}(\lambda - \mu)c_1^2, \quad (23)$$

where  $\lambda$  is defined in (18).

Applying triangular inequality in (23), it yields

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(A-B)p^2|c_2|}{2(p+2\alpha)} + \frac{(A-B)^2p^4}{(p+\alpha)^2}|\lambda - \mu||c_1|^2. \quad (24)$$

Using (2) in (24), we obtain

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(A-B)p^2}{2(p+2\alpha)} + \frac{(A-B)^2 p^4}{(p+\alpha)^2} [|\lambda - \mu| - \nu], \quad (25)$$

where  $\nu$  is defined in (19).

**Case I.**  $\mu \leq \lambda$ .

From (2.9), we have

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(A-B)p^2}{2(p+2\alpha)} + \frac{(A-B)^2 p^4}{(p+\alpha)^2} [(\lambda - \nu) - \mu]. \quad (26)$$

If  $\mu \leq (\lambda - \nu)$ , from (26)

$$\begin{aligned} |a_{p+2} - \mu a_{p+1}^2| &\leq \frac{(A-B)p^2}{2(p+2\alpha)} + \frac{(A-B)^2 p^4}{(p+\alpha)^2} (\lambda - \mu - \nu) \\ &= \frac{(A-B)^2 p^4}{(p+\alpha)^2} (\lambda - \mu). \end{aligned}$$

If  $\lambda - \nu \leq \mu \leq \lambda$ , again from (26)

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(A-B)p^2}{2(p+2\alpha)}.$$

**Case II.**  $\mu \geq \lambda$ .

From (2.9), we have

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(A-B)p^2}{2(p+2\alpha)} + \frac{(A-B)^2 p^4}{(p+\alpha)^2} [\mu - (\lambda + \nu)]. \quad (27)$$

If  $\mu \leq (\lambda + \nu)$ , from (27)

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(A-B)p^2}{2(p+2\alpha)}.$$

If  $\mu \geq (\lambda + \nu)$ , again from (27)

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(A-B)^2 p^4}{(p+\alpha)^2} (\mu - \lambda).$$

For  $\alpha = 0$ , equality sign hold in first and third inequalities of (17) for the function  $f_1(z)$  defined by

$$f_1(z) = \begin{cases} z^p(1 + B\delta z)^{\frac{p(A-B)}{B}}, & B \neq 0; \\ z^p e^{(pA\delta z)}, & B = 0, |\delta| = 1. \end{cases} \quad (28)$$

and equality sign hold in second inequality of (17) for the function  $f_0(z)$  defined by

$$f_0(z) = \begin{cases} z^p(1 + B\delta z^2)^{\frac{p(A-B)}{2B}}, & B \neq 0; \\ z^p e^{\frac{(pA\delta z^2)}{2}}, & B = 0, |\delta| = 1. \end{cases} \quad (29)$$

For  $\alpha > 0$ , equality sign hold in first and third inequalities of (17) for the function  $f_{1(\alpha)}(z)$  defined by

$$f_{1(\alpha)}(z) = \begin{cases} \left[ \frac{1}{\alpha} \left( \int_0^z t^{\frac{p}{\alpha}-1} (1 + B\delta t)^{\frac{p(A-B)}{B\alpha}} dt \right) \right]^\alpha, & B \neq 0; \\ \left[ \frac{1}{\alpha} \left( \int_0^z t^{\frac{p}{\alpha}-1} e^{\frac{(pA\delta t)}{\alpha}} dt \right) \right]^\alpha, & B = 0, |\delta| = 1. \end{cases} \quad (30)$$

and equality sign hold in second inequality of (17) for the function  $f_{0(\alpha)}(z)$  defined by

$$f_{0(\alpha)}(z) = \begin{cases} \left[ \frac{1}{\alpha} \left( \int_0^z t^{\frac{p}{\alpha}-1} (1 + B\delta t^2)^{\frac{p(A-B)}{2B\alpha}} dt \right) \right]^\alpha, & B \neq 0; \\ \left[ \frac{1}{\alpha} \left( \int_0^z t^{\frac{p}{\alpha}-1} e^{\frac{(pA\delta t^2)}{2\alpha}} dt \right) \right]^\alpha, & B = 0, |\delta| = 1. \end{cases} \quad (31)$$

**Remark 2.1.** Choosing  $p = 1$  in the Theorem 2.1, we get the results proved by Goel and Mehrok [4].

Choosing  $A = 1, B = -1$  in the Theorem 2.1, we have

**Corollary 2.1.** If  $f \in M_p(\alpha)$ , then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} \frac{4p^4}{(p+\alpha)^2} (\lambda_1 - \mu), & \mu \leq \lambda_1 - \nu_1; \\ \frac{p^2}{p+2\alpha}, & \lambda_1 - \nu_1 \leq \mu \leq \lambda_1 + \nu_1; \\ \frac{4p^4}{(p+\alpha)^2} (\mu - \lambda_1), & \mu \geq \lambda_1 + \nu_1. \end{cases} \quad (32)$$

where

$$\lambda_1 = \frac{2p(p^2 + 2p\alpha + \alpha) + (p + \alpha)^2}{4p^2(p + 2\alpha)}$$

and

$$\nu_1 = \frac{(p + \alpha)^2}{4p^2(p + 2\alpha)}.$$

For  $\alpha = 0$ , the extremal function for the first and third inequalities of (32) is obtained by putting  $A = 1, B = -1$  in (28) and extremal function for the second inequality of (30) is obtained by putting  $A = 1, B = -1$  in (29).

For  $\alpha > 0$ , the extremal function for the first and third inequalities of (32) is obtained by putting  $A = 1, B = -1$  in (30) and extremal function for the second inequality of (32) is obtained by putting  $A = 1, B = -1$  in (31).

Choosing  $\alpha = 0$  in the Theorem 2.1, we have

**Corollary 2.2.** If  $f \in S_p^*(A, B)$ , then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} \frac{p(A-B)}{2} [(A-B)p - B] - p^2(A-B)^2\mu, & \mu \leq \frac{(A-B)p - (1+B)}{2(A-B)p}; \\ \frac{p(A-B)}{2}, \frac{(A-B)p - (1+B)}{2(A-B)p} \leq \mu \leq \frac{(A-B)p + (1-B)}{2(A-B)p}; \\ p^2(A-B)^2\mu - \frac{p(A-B)}{2} [(A-B)p - B], & \mu \geq \frac{(A-B)p + (1-B)}{2(A-B)p}. \end{cases} \quad (33)$$

The first and third inequalities of (33) are sharp for the function  $f_1(z)$  defined in (28) and the second inequality is sharp for the function  $f_0(z)$  defined in (29).

**Remark 2.2.** By taking  $p = 1$  in the Corollary 2.2, we get the estimates of  $|a_3 - \mu a_2^2|$  for the class  $S^*(A, B)$ .

**Remark 2.3.** Taking  $A = 1, B = -1$  in the Corollary 2.2, we obtain the following result: If  $f \in S_p^*$ , then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} p(2p+1) - 4p^2\mu, & \mu \leq \frac{1}{2}; \\ p, & \frac{1}{2} \leq \mu \leq \frac{p+1}{2p}; \\ 4p^2\mu - p(2p+1), & \mu \geq \frac{p+1}{2p}. \end{cases}$$

**Remark 2.4.** Taking  $p = 1, A = 1, B = -1$  in the Corollary 2.2, we obtain the following result: If  $f \in S^*$ , then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \mu \leq \frac{1}{2}; \\ 1, & \frac{1}{2} \leq \mu \leq 1; \\ 4\mu - 3, & \mu \geq 1. \end{cases}$$

This result is due to Keogh and Merkes [7].

Putting  $\alpha = 1$  in the Theorem 2.1, we get

**Corollary 2.3.** If  $f \in K_p(A, B)$ , then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} \frac{(A-B)[(A-B)p-B]p^2}{2(p+2)} - \frac{(A-B)^2p^4}{(p+1)^2}\mu, & \mu \leq \frac{(p+1)^2[(A-B)p-(1+B)]}{2(A-B)p^2(p+2)}; \\ \frac{(A-B)p^2}{2(p+2)}, \frac{(p+1)^2[(A-B)p-(B+1)]}{2(A-B)p^2(p+2)} \leq \mu \leq \frac{(p+1)^2[(A-B)p+(1-B)]}{2(A-B)p^2(p+2)}; \\ \frac{(A-B)^2p^4}{(p+1)^2}\mu - \frac{(A-B)[(A-B)p-B]p^2}{2(p+2)}, & \mu \geq \frac{(p+1)^2[(A-B)p+(1-B)]}{2(A-B)p^2(p+2)}. \end{cases} \quad (34)$$

The first and third inequalities of (34) are sharp for the function defined by

$$f_1(z) = \begin{cases} \int_0^z pt^{p-1}(1+B\delta t)^{\frac{p(A-B)}{B}} dt, & B \neq 0; \\ \int_0^z pt^{p-1}e^{pA\delta t} dt, & B = 0, |\delta| = 1. \end{cases}$$

and the second inequality of (34) is sharp for the function defined by

$$f_0(z) = \begin{cases} \int_0^z pt^{p-1}(1+B\delta t^2)^{\frac{p(A-B)}{2B}} dt, & B \neq 0; \\ \int_0^z pt^{p-1}e^{\frac{pA\delta t^2}{2}} dt, & B = 0, |\delta| = 1. \end{cases}$$

**Remark 2.5.** Letting  $p = 1$  in Corollary 2.3, we obtain the estimates of  $|a_3 - \mu a_2^2|$  for the class  $K(A, B)$ .

**Remark 2.6.**



Letting  $A = 1, B = -1$  in Corollary 2.3, we have the result: If  $f \in K_p$ , then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} \frac{p^2(2p+1)}{p+2} - \frac{4p^4}{(p+1)^2} \mu, \mu \leq \frac{(p+1)^2}{2p(p+2)}; \\ \frac{p^2}{p+2}, \frac{(p+1)^2}{2p(p+2)} \leq \mu \leq \frac{(p+1)^3}{2p^2(p+2)}; \\ \frac{4p^4}{(p+1)^2} \mu - \frac{p^2(2p+1)}{p+2}, \mu \geq \frac{(p+1)^3}{2p^2(p+2)}. \end{cases}$$

**Remark 2.7.** Taking  $p = 1, A = 1, B = -1$  in the Corollary 2.3, we obtain the following result: If  $f \in K$ , then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1 - \mu, \mu \leq \frac{2}{3}; \\ \frac{1}{3}, \frac{2}{3} \leq \mu \leq \frac{4}{3}; \\ \mu - 1, \mu \geq \frac{4}{3}. \end{cases}$$

This result is due to Keogh and Merkes [7].

## References

- [1] M. Fekete and G. Szegő, Eine Bemerkung über ungerade schlichte functionen, J. London Math. Soc., **8**(1933), 85-89.
- [2] R. M. Goel and B. S. Mehrok, On the coefficients of a subclass of starlike functions, Ind. J. Pure Appl. Math., **12**(1981), 634-647.
- [3] R. M. Goel and B. S. Mehrok, On a class of close to convex functions, Ind. J. Pure Appl. Math., **12**(1981), 648-658.
- [4] R. M. Goel and B. S. Mehrok, A coefficient inequality for certain classes of analytic functions, Tamkang J. Math., **22**(1991), 153-163.
- [5] R. M. Goel and B. S. Mehrok, On the order of starlikeness of a subclass of  $\alpha$  convex functions, Soochow J. Math., **17**(1991), 55-65.
- [6] W. Janowski, Some extremal problems for certain families of analytic functions, Ann. Polon. Math., **28**(1973), 297-326.
- [7] S. R. Keogh and E. P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc., **20**(1969), 8-12.
- [8] S. S. Miller, P. T. Mocanu and M. O. Reade, All  $\alpha$  convex functions are univalent and starlike, Proc. Amer. Math. Soc., **37**(1973), 553-554.
- [9] P. T. Mocanu, Une propriete' de convexite' generalisee dans la theorie de la representation conforme, Mathematica(CLUJ), **34**(1969), 127-133.

# The mean value of function $q_k^{(e)}(n)$ over cube-full number

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**Abstract** Let  $n > 1$  be an integer,  $q_k^{(e)}(n)$  is a multiplicative function. In this paper, we shall study the mean value of exponential divisor function  $q_k^{(e)}(n)$  over cube-full number, that is

$$\sum_{\substack{n \leq x \\ n \text{ is cube-full}}} q_k^{(e)}(n) = \sum_{n \leq x} q_k^{(e)}(n) f_3(n),$$

where  $f_3(n)$  is the characteristic function of cube-full integers, i.e.

$$f_3(n) = \begin{cases} 1, & n \text{ is cube-full;} \\ 0, & \text{otherwise.} \end{cases}$$

**Keywords** cube-full number, divisor problem, Dirichlet convolution, residue theorem, mean value.

## §1. Introduction

Let  $n > 1$  be an integer of canonical form  $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ . The integer  $n$  is called a  $k$ -full number if  $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ , where  $a_1 \geq k, a_2 \geq k, \dots, a_r \geq k$ . Let  $f_k(n)$  be the characteristic function of  $k$ -full integers, i.e.

$$f_k(n) = \begin{cases} 1, & n \text{ is } k\text{-full;} \\ 0, & \text{otherwise.} \end{cases}$$

We call an integer  $n = \prod_{i=1}^r p_i^{a_i}$  exponentially  $k$ -free if all the exponents  $a_i$  ( $1 \leq i \leq r$ ) are  $k$ -free, i.e. are not divisible by the  $k$ -th power of any prime ( $k \geq 2$ ). Let  $q_k^{(e)}(n)$  denote the characteristic function of exponentially  $k$ -free integers.

Obviously the function  $q_k^{(e)}(n)$  is multiplicative, and for every prime power  $p^a$ , there are  $q_k^{(e)}(p) = q_k^{(e)}(p^2) = q_k^{(e)}(p^3) = \dots = q_k^{(e)}(p^{2^k-1}) = 1, q_k^{(e)}(p^{2^k}) = 0$ .

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Many authors have studied the properties of the exponential divisor function  $q_k^{(e)}(n)$ . L. Toth [2] proved the following result:

$$\sum_{n \leq x} q_k^{(e)}(n) = D_k x + O(x^{1/2^k} \delta(x)),$$

where

$$D_k = \prod_p \left( 1 + \sum_{a=2^k}^{\infty} \frac{q_k(a) - q_k(a-1)}{p^a} \right),$$

$q_k(n)$  denoting the characteristic function of  $k$ -free integers.

In this paper, we shall study the mean value of exponential divisor function  $q_k^{(e)}(n)$  over cube-full number, that is

$$\sum_{\substack{n \leq x \\ n \text{ is cube-full}}} q_k^{(e)}(n) = \sum_{n \leq x} q_k^{(e)}(n) f_3(n),$$

where  $f_3(n)$  is the characteristic function of cube-full integers, i.e.

$$f_3(n) = \begin{cases} 1, & n \text{ is cube-full;} \\ 0, & \text{otherwise.} \end{cases}$$

**Theorem 1.1.** For some  $D > 0$ ,

$$\begin{aligned} \sum_{\substack{n \leq x \\ n \text{ is cube-full}}} q_k^{(e)}(n) &= \frac{\zeta(\frac{4}{3})\zeta(\frac{5}{3})G(\frac{1}{3})}{\zeta(\frac{8}{3})} x^{\frac{1}{3}} + \frac{\zeta(\frac{3}{4})\zeta(\frac{5}{4})G(\frac{1}{4})}{\zeta(2)} x^{\frac{1}{4}} + \frac{\zeta(\frac{3}{5})\zeta(\frac{4}{5})G(\frac{1}{5})}{\zeta(\frac{8}{5})} x^{\frac{1}{5}} \\ &\quad + O\left(x^{\frac{1}{8}} \exp\left(-D(\log x)^{\frac{3}{5}}(\log \log x)^{-\frac{1}{5}}\right)\right), \end{aligned}$$

where  $G(s) = \prod_p \left(1 - \frac{1}{p^{9s}} + \dots\right)$  is absolutely convergent for  $\text{Res} > \frac{1}{9}$ .

**Notations 1.1.** Throughout this paper,  $\epsilon$  always denotes a fixed but sufficiently small positive constant.

## §2. Proof of the theorem

In order to prove our theorem, we need the following lemmas.

**Lemma 2.1.** [3]

$$\Delta(3, 4, 5; x) \ll x^{\frac{22}{177}} \log^3 x.$$

**Proof.** The proof of this bound depends on the theory of two-dimensional exponent pairs.

**Lemma 2.2.** Suppose  $f(n)$  is arithmetical function, and satisfy

$$\begin{aligned} \sum_{n \leq x} f(n) &= \sum_{j=1}^l x^{a_j} P_j(\log x) + O(x^a), \\ \sum_{n \leq x} |f(n)| &= O(x^{a_1} \log^r x), \end{aligned}$$

here  $a_1 \geq a_2 \geq \cdots \geq a_l > 1/c > a \geq 0, r \geq 0, P_1(t), \dots, P_l(t)$  are polynomials on  $t$  whose degree are not exceed  $r$ , and  $c \geq 1, b \geq 1$  are fixed integers.

Suppose for  $Res > 1$ , have

$$\sum_{n=1}^{\infty} \frac{\mu_b(n)}{n^s} = \frac{1}{\zeta^b(s)}.$$

If

$$h(n) = \sum_{d^c | n} \mu_b(d) f(n/d^c),$$

then

$$\sum_{n \leq x} h(n) = \sum_{j=1}^l x^{a_j} R_j(\log x) + E_c(x),$$

here  $R_1(t), \dots, R_l(t)$  are polynomials on  $t$  whose degree are not exceed  $r$ , and for some  $D > 0$ , have

$$E_c(x) \ll x^{1/c} \exp\left((-D(\log x)^{3/5}(\log \log x)^{-1/5})\right).$$

**Proof.** If  $b = 1$ , see theorem 14.2 in A. Ivić [2]. If  $b \geq 2$ , it can be proved in the same method.

**Lemma 2.3.** Let  $s = \sigma + it$  is a complex number, then we have

$$\sum_{\substack{n=1 \\ n \text{ is cube-full}}}^{\infty} \frac{q_k^{(e)}(n)}{n^s} = \frac{\zeta(3s)\zeta(4s)\zeta(5s)}{\zeta(8s)} G(s), \quad (1)$$

where the Dirichlet series  $G(s) = \prod_p \left(1 - \frac{1}{p^{9s}} + \cdots\right)$  is absolutely convergent for  $Res > \frac{1}{9}$ .

**Proof.** Let

$$F(s) := \sum_{\substack{n=1 \\ n \text{ is cube-full}}}^{\infty} \frac{q_k^{(e)}(n)}{n^s} = \sum_{n=1}^{\infty} \frac{q_k^{(e)}(n) f_3(n)}{n^s}, (Res > 1)$$

here

$$f_3(n) = \begin{cases} 1, & n \text{ is cube-full;} \\ 0, & \text{otherwise.} \end{cases}$$

From  $q_k^{(e)}(n)$  is multiplicative we have for  $Res > 1$  that,

$$\begin{aligned}
F(s) &= \sum_{\substack{n=1 \\ n \text{ is cube-full}}}^{\infty} \frac{q_k^{(e)}(n)}{n^s} = \sum_{n=1}^{\infty} \frac{q_k^{(e)}(n)f_3(n)}{n^s} \\
&= \prod_p \left( 1 + \frac{q_k^{(e)}(p)f_3(p)}{p^s} + \frac{q_k^{(e)}(p^2)f_3(p^2)}{p^{2s}} + \frac{q_k^{(e)}(p^3)f_3(p^3)}{p^{3s}} + \cdots + \frac{q_k^{(e)}(p^{2^k-1})f_3(p^{2^k-1})}{p^{(2^k-1)s}} \right) \\
&= \prod_p \left( 1 + \frac{1}{p^{3s}} + \frac{1}{p^{4s}} + \frac{1}{p^{5s}} + \cdots + \frac{1}{p^{(2^k-1)s}} \right) \\
&= \zeta(3s) \prod_p \left( 1 + \frac{1}{p^{4s}} + \frac{1}{p^{5s}} - \frac{1}{p^{2^k s}} - \frac{1}{p^{(2^k+1)s}} - \frac{1}{p^{(2^k+2)s}} \right) \\
&= \zeta(3s)\zeta(4s) \prod_p \left( 1 + \frac{1}{p^{5s}} - \frac{1}{p^{8s}} - \frac{1}{p^{9s}} - \frac{1}{p^{2^k s}} - \frac{1}{p^{(2^k+1)s}} + \cdots \right) \\
&= \zeta(3s)\zeta(4s)\zeta(5s) \prod_p \left( 1 - \frac{1}{p^{8s}} - \frac{1}{p^{9s}} + \cdots \right) \\
&= \frac{\zeta(3s)\zeta(4s)\zeta(5s)}{\zeta(8s)} \prod_p \left( 1 - \frac{1}{p^{9s}} + \cdots \right) \\
&= \frac{\zeta(3s)\zeta(4s)\zeta(5s)}{\zeta(8s)} G(s),
\end{aligned}$$

where  $G(s) = \prod_p \left( 1 - \frac{1}{p^{9s}} + \cdots \right)$ , and it is absolutely convergent for  $\sigma > \frac{1}{9} + \epsilon$ .

Next we prove our theorem.

From lemma 2.3, we have known

$$F(s) := \sum_{\substack{n=1 \\ n \text{ is cube-full}}}^{\infty} \frac{q_k^{(e)}(n)}{n^s} = \frac{\zeta(3s)\zeta(4s)\zeta(5s)}{\zeta(8s)} G(s),$$

where  $G(s)$  is absolutely convergent for  $\sigma > \frac{1}{9} + \epsilon$ .

Define

$$\begin{aligned}
G(s) &:= \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \\
H(s) &:= \zeta(3s)\zeta(4s)\zeta(5s)G(s) := \sum_{n=1}^{\infty} \frac{h(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\sum_{m|n} d(3, 4, 5; m)g(l)}{n^s}, (Res > 1)
\end{aligned}$$

here  $h(n) = \sum_{m|n} d(3, 4, 5; m)g(l)$ . Then we can get

$$\sum_{n \leq x} h(n) = \sum_{ml \leq x} d(3, 4, 5; m)g(l) = \sum_{l \leq x} g(l) \sum_{m \leq \frac{x}{l}} d(3, 4, 5; m).$$

From Residue theorem and lemma 2.1, we can get

$$\begin{aligned}
\sum_{n \leq x} d(3, 4, 5; n) &= \zeta\left(\frac{4}{3}\right)\zeta\left(\frac{5}{3}\right)x^{\frac{1}{3}} + \zeta\left(\frac{3}{4}\right)\zeta\left(\frac{5}{4}\right)x^{\frac{1}{4}} + \zeta\left(\frac{3}{5}\right)\zeta\left(\frac{4}{5}\right)x^{\frac{1}{5}} + \Delta(3, 4, 5; x) \\
&= \zeta\left(\frac{4}{3}\right)\zeta\left(\frac{5}{3}\right)x^{\frac{1}{3}} + \zeta\left(\frac{3}{4}\right)\zeta\left(\frac{5}{4}\right)x^{\frac{1}{4}} + \zeta\left(\frac{3}{5}\right)\zeta\left(\frac{4}{5}\right)x^{\frac{1}{5}} + O(x^{\frac{22}{177}}). \tag{2}
\end{aligned}$$

Then from (2) and Abel integral formula, we have the relation:

$$\begin{aligned}
 \sum_{n \leq x} h(n) &= \sum_{l \leq x} g(l) \left[ \zeta\left(\frac{4}{3}\right) \zeta\left(\frac{5}{3}\right) \left(\frac{x}{l}\right)^{\frac{1}{3}} + \zeta\left(\frac{3}{4}\right) \zeta\left(\frac{5}{4}\right) \left(\frac{x}{l}\right)^{\frac{1}{4}} + \zeta\left(\frac{3}{5}\right) \zeta\left(\frac{4}{5}\right) \left(\frac{x}{l}\right)^{\frac{1}{5}} + O\left(\left(\frac{x}{l}\right)^{\frac{22}{177}}\right) \right] \\
 &= \zeta\left(\frac{4}{3}\right) \zeta\left(\frac{5}{3}\right) x^{\frac{1}{3}} \sum_{l \leq x} \frac{g(l)}{l^{1/3}} + \zeta\left(\frac{3}{4}\right) \zeta\left(\frac{5}{4}\right) x^{\frac{1}{4}} \sum_{l \leq x} \frac{g(l)}{l^{1/4}} + \zeta\left(\frac{3}{5}\right) \zeta\left(\frac{4}{5}\right) x^{\frac{1}{5}} \sum_{l \leq x} \frac{g(l)}{l^{1/5}} + O\left(x^{\frac{22}{177}} \sum_{l \leq x} \frac{|g(l)|}{l^{22/177}}\right) \\
 &= \zeta\left(\frac{4}{3}\right) \zeta\left(\frac{5}{3}\right) x^{\frac{1}{3}} \sum_{l=1}^{\infty} \frac{g(l)}{l^{1/3}} + \zeta\left(\frac{3}{4}\right) \zeta\left(\frac{5}{4}\right) x^{\frac{1}{4}} \sum_{l=1}^{\infty} \frac{g(l)}{l^{1/4}} + \zeta\left(\frac{3}{5}\right) \zeta\left(\frac{4}{5}\right) x^{\frac{1}{5}} \sum_{l=1}^{\infty} \frac{g(l)}{l^{1/5}} + O\left(x^{\frac{1}{3}} \sum_{l > x} \frac{|g(l)|}{l^{1/3}}\right) \\
 &\quad + O\left(x^{\frac{1}{4}} \sum_{l > x} \frac{|g(l)|}{l^{1/4}}\right) + O\left(x^{\frac{1}{5}} \sum_{l > x} \frac{|g(l)|}{l^{1/5}}\right) + O\left(x^{\frac{22}{177}} \sum_{l \leq x} \frac{|g(l)|}{l^{22/177}}\right) \\
 &= \zeta\left(\frac{4}{3}\right) \zeta\left(\frac{5}{3}\right) G\left(\frac{1}{3}\right) x^{\frac{1}{3}} + \zeta\left(\frac{3}{4}\right) \zeta\left(\frac{5}{4}\right) G\left(\frac{1}{4}\right) x^{\frac{1}{4}} + \zeta\left(\frac{3}{5}\right) \zeta\left(\frac{4}{5}\right) G\left(\frac{1}{5}\right) x^{\frac{1}{5}} + O(x^{\frac{1}{9}}).
 \end{aligned}$$

From lemma 2.2 and Perron's formula [4], we can get:

$$\begin{aligned}
 \sum_{\substack{n \leq x \\ n \text{ is cube-full}}} q_k^{(e)}(n) &= \frac{\zeta\left(\frac{4}{3}\right) \zeta\left(\frac{5}{3}\right) G\left(\frac{1}{3}\right)}{\zeta\left(\frac{8}{3}\right)} x^{\frac{1}{3}} + \frac{\zeta\left(\frac{3}{4}\right) \zeta\left(\frac{5}{4}\right) G\left(\frac{1}{4}\right)}{\zeta(2)} x^{\frac{1}{4}} + \frac{\zeta\left(\frac{3}{5}\right) \zeta\left(\frac{4}{5}\right) G\left(\frac{1}{5}\right)}{\zeta\left(\frac{8}{5}\right)} x^{\frac{1}{5}} \\
 &\quad + O\left(x^{\frac{1}{8}} \exp\left(-D(\log x)^{\frac{3}{5}} (\log \log x)^{-\frac{1}{5}}\right)\right).
 \end{aligned}$$

Here  $D > 0$ , and  $G(s) = \prod_p \left(1 - \frac{1}{p^s} + \dots\right)$  is absolutely convergent for  $\text{Res} > \frac{1}{9}$ .

Now our theorem is proved.

## References

- [1] M. V. Subbarao, On some arithmetic convolutions, in the Theory of Arithmetic Functions, lecture notes in mathematics, Vol. 251, Springer, (1972), 247-271.
- [2] L. Tóth, On certain arithmetical functions involving exponential divisors, Ann. Univ. Sci. Budapest. Sect. Comput, **24**(2004), 285-294.
- [3] A. Ivić, The Riemann Zeta-function, John Wiley & Sons, 1985.
- [4] E. C. Titchmarsh, The theory of the Riemann Zeta-function (revised by D. R. Heath-Brown), Oxford university Press, 1986.
- [5] L. Tóth, On certain arithmetical functions involving exponential divisors II, Ann. Univ. Sci. Budapest. Sect. Comput, **27**(2007), 155-166.

# Some arithmetical properties of the Smarandache series

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**Abstract** The Smarandache function  $S(n)$  is defined as the minimal positive integer  $m$  such that  $n|m!$ . The main purpose of this paper is to study the analyze converges questions for some series of the form  $\sum_{n=1}^{\infty} \frac{1}{S(n)^\delta}$ , i.e., we proved the series  $\sum_{n=1}^{\infty} \frac{1}{S(n)^\delta}$  diverges for any  $\delta \leq 1$ , and  $\sum_{n=1}^{\infty} \frac{1}{S(n)^{\epsilon S(n)}}$  converges for any  $\epsilon > 0$ .

**Keywords** Smarandache function, smarandache series, converges.

## §1. Introduction and results

For every positive integer  $n$ , let  $S(n)$  be the minimal positive integer  $m$  such that  $n|m!$ , i.e.,

$$S(n) = \min\{m : m \in \mathbb{N}, n|m!\}.$$

This function is known as Smarandache function [1]. Easily, one has  $S(1) = 1, S(2) = 2, S(3) = 3, S(4) = 4, S(5) = 5, S(6) = 3, S(7) = 7, S(8) = 4, S(9) = 6, S(10) = 5, \dots$ .

Use the standard factorization of  $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ ,  $p_1 < p_2 < \cdots < p_k$ , it's trivial to have

$$S(n) = \max_{1 \leq i \leq k} \{S(p_i^{\alpha_i})\}.$$

Many scholars have studied the properties of  $S(n)$ , for example, M. Farris and P. Mitchell [2] show the boundary of  $S(p^\alpha)$  as

$$(p-1)\alpha + 1 \leq S(p^\alpha) \leq (p-1)[\alpha + 1 + \log_p \alpha] + 1.$$

Z. Xu [3] noticed the following interesting relationship formula

$$\pi(x) = -1 + \sum_{n=2}^{[x]} \left[ \frac{S(n)}{n} \right],$$

by the fact that  $S(p) = p$  for  $p$  prime and  $S(n) < n$  except for the case  $n = 4$  and  $n = p$ , where  $\pi(x)$  denotes the number of prime up to  $x$ , and  $[x]$  the greatest integer less or equal to  $x$ . Those and many other interesting results on Smarandache function  $S(n)$ , readers may refer to [2]-[6].

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Let  $p$  be a fixed prime and  $n \in \mathbb{N}$ , the primitive numbers of power  $p$ , denoted by  $S_p(n)$ , is defined by

$$S_p(n) = \min\{m : m \in \mathbb{N}, p^n | m!\} = S(p^n).$$

Z. Xu <sup>[3]</sup> obtained the identity between Riemann zeta function  $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ ,  $\sigma > 1$  and an infinite series involving  $S_p(n)$  as

$$\sum_{n=1}^{\infty} \frac{1}{S_p^s(n)} = \frac{\zeta(s)}{p^s - 1},$$

and he also obtained some other asymptotic formulae for  $S_p(n)$ . F. Luca <sup>[4]</sup> proved the series  $\sum_{n=1}^{\infty} \frac{1}{S(n)^{S(n)^\delta}}$  converges for all  $\delta \geq 1$  and diverges for all  $\delta < 1$ , and the series  $\sum_{n=1}^{\infty} \frac{1}{S(n)^{\epsilon \log n}}$  converges for any  $\epsilon > 0$ .

In this note, we studied the analyze converges problems for the infinite series involving  $S(n)$ . That is, we shall prove the following conclusions:

**Theorem 1.1.** For any  $\delta \leq 1$ , the series

$$\sum_{n=1}^{\infty} \frac{1}{S(n)^\delta}$$

diverges.

**Theorem 1.2.** For any  $\epsilon > 0$ , the series

$$\sum_{n=1}^{\infty} \frac{1}{S(n)^{\epsilon S(n)}}$$

converges.

## §2. Some lemmas

To complete the proof of theorems, we need two Lemmas.

**Lemma 2.1.** Let  $p$  be any fixed prime. Then for any real number  $x \geq 1$ , we have the asymptotic formula:

$$\sum_{\substack{n=1 \\ S_p(n) \leq x}}^{\infty} \frac{1}{S_p(n)} = \frac{1}{p-1} \left( \ln x + \gamma + \frac{p \ln p}{p-1} \right) + O(x^{-\frac{1}{2}} + \epsilon),$$

where  $\gamma$  is the Euler constant,  $\epsilon$  denotes any fixed positive numbers.

**Proof.** See Theorem 2 of [3].

**Lemma 2.2.**<sup>[7]</sup> Let  $\epsilon > 0$  and  $d(n)$  denotes the divisor function of positive integer  $n$ . Then

$$d(n) = O(n^\epsilon) \leq C_\epsilon n^\epsilon,$$

where the o-constant  $C_\epsilon$  depends on  $\epsilon$ .

**Proof.** The proof follows [7] by writing  $n = \prod_{p|n} p^\alpha$ , the standard factorization of  $n$ . Then

$$p^{\alpha\epsilon} \geq 2^{\alpha\epsilon} = e^{\alpha\epsilon \ln 2} \geq \alpha\epsilon \ln 2 \geq \frac{1}{2}(a+1)\epsilon \ln 2.$$



If  $p^\epsilon \geq 2$ , then  $p^{\alpha\epsilon} \geq 2^\alpha \geq \alpha + 1$ . Therefore,

$$\frac{d(n)}{n^\epsilon} = \prod_{p|n} \frac{\alpha + 1}{p^{\alpha\epsilon}} = \prod_{\substack{p|n \\ p^\epsilon < 2}} \frac{\alpha + 1}{p^{\alpha\epsilon}} \prod_{\substack{p|n \\ p^\epsilon \geq 2}} \frac{\alpha + 1}{p^{\alpha\epsilon}} \geq \prod_{\substack{p|n \\ p^\epsilon < 2}} \frac{\alpha + 1}{\frac{1}{2}(a+1)\epsilon \ln 2} \prod_{\substack{p|n \\ p^\epsilon \geq 2}} \frac{\alpha + 1}{\alpha + 1}.$$

The last item in above inequality is  $\prod_{\substack{p|n \\ p^\epsilon < 2}} \frac{2}{\epsilon \ln 2}$ , which is less than  $\prod_{p^\epsilon < 2} \frac{2}{\epsilon \ln 2} = C_\epsilon$ , say, the o-constant  $C_\epsilon$  depends on  $\epsilon$ .

### §3. Proof of theorems

#### Proof of Theorem 1.

We may treat the case  $\delta = 1$  first. By Lemma 1 and the notation  $S_p(n) = S(p^n)$ , we have

$$\sum_{n=1}^{\infty} \frac{1}{S(p^n)} = \lim_{x \rightarrow +\infty} \sum_{\substack{n=1 \\ S_p(n) \leq x}}^{\infty} \frac{1}{S_p(n)} = \infty.$$

Obviously, for  $\delta \leq 1$ ,  $\sum_{n=1}^{\infty} \frac{1}{S(n)^\delta}$  diverges follows easily by the trivial inequality:

$$\sum_{n=1}^{\infty} \frac{1}{S(n)^\delta} \geq \sum_{n=1}^{\infty} \frac{1}{S(n)} \geq \sum_{n=1}^{\infty} \frac{1}{S(p^n)},$$

complete the proof.

#### Proof of Theorem 2.

It certainly suffices to assume that  $\epsilon \leq 1$ . We rewrite series  $\sum_{n=1}^{\infty} \frac{1}{S(n)^{\epsilon S(n)}}$  as

$$\sum_{k=1}^{\infty} \frac{u(k)}{k^{\epsilon k}},$$

where  $u(k) = \#\{n : S(n) = k\}$ . For every positive integer  $n$  such that  $S(n) = k$  is a divisor of  $k!$ , i.e.  $u(k) \leq d(k!)$ . By Lemma 2 and the inequality bellow

$$(k!)^2 = \prod_{j=1}^k j(k+1-j) < \prod_{j=1}^k \left(\frac{k+1}{2}\right)^2 = \left(\frac{k+1}{2}\right)^{2k}.$$

we have

$$u(k) \leq d(k!) \leq C_\epsilon (k!)^\epsilon < C_\epsilon \left(\frac{k+1}{2}\right)^{\epsilon k}.$$

where  $C_\epsilon$  means that the constant depending on  $\epsilon$ .

Therefore, recalling that the properties of the sequence  $(1 + \frac{1}{k})^k$ , we have

$$\sum_{k=1}^{\infty} \frac{u(k)}{k^{\epsilon k}} \leq C_\epsilon \sum_{k=1}^{\infty} \frac{1}{k^{\epsilon k}} \left(\frac{k+1}{2}\right)^{\epsilon k} = C_\epsilon \sum_{k=1}^{\infty} \frac{1}{2^{\epsilon k}} \left(\frac{k+1}{k}\right)^{\epsilon k} < C_1 \sum_{k=1}^{\infty} \frac{1}{2^{\epsilon k}},$$

for some constant  $C_1$ , it follows that series  $\sum_{k=1}^{\infty} \frac{u(k)}{k^\varepsilon}$  is bounded above by

$$C_1 \sum_{k=1}^{\infty} \frac{1}{2^{\varepsilon k}} = \frac{C_1}{2^\varepsilon - 1},$$

completing the proof.

## References

- [1] F. Smarandache, Only problem not solution, Xiquan Publishing House, Chicago, 1993.
- [2] M. Farris and P. Mitchell, Bounding the Smarandache function, Smarandache Notation Journal, **13**(2002), 37-42.
- [3] Z. Xu, Some arithmetical properties of primitive numbers of power  $p$ , **2**(2006), No. 1, 9-12.
- [4] F. Luca, The Smarandache harmonic series, NAW, **5**(2000), No. 1, 150-151.
- [5] Jozsef Sandor, On certain inequalities involving the Smarandache dunction, Scientia Magna, **2**(2006), No. 3, 78-80.
- [6] J. Fu, An equaltion involving the Smarandache function, Scientia Magna, **2**(2006), No. 4, 83-86.
- [7] L. K. Hua, Itroduction to number theory, Springer-Verlag, 1982.

# Transmuted Weibull-geometric distribution and its applications

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**Abstract** A generalization of the Weibull geometric distribution so-called the transmuted Weibull geometric distribution is proposed and studied. Various structural properties including explicit expressions for the moments, moment generating function of the new distribution are derived. The estimation of the model parameters is performed by maximum likelihood method. We hope that the new distribution proposed here will serve as an alternative model to the other models which are available in the literature for modeling positive real data in many areas.

**Keywords** Weibull-geometric distribution, moments, order statistics, transmutation map, maximum likelihood estimation, reliability function.

**2000 Mathematics Subject Classification:** 62E15, 62F30.

## §1. Introduction and motivation

The Weibull distribution is one of the most commonly used lifetime distribution in modeling lifetime data. In practice, it has been shown to be very flexible in modeling various types of lifetime distributions with monotone failure rates but it is not useful for modeling the bathtub shaped and the unimodal failure rates which are common in reliability and biological studies. Several distributions have been proposed in the literature to model lifetime data. Adamidis and Loukas <sup>[2]</sup> introduced the two-parameter exponential-geometric (*EG*) distribution with decreasing failure rate. Kus <sup>[9]</sup> introduced the exponential-Poisson distribution (following the same idea of the *EG* distribution) with decreasing failure rate and discussed various of its properties. Marshall and Olkin <sup>[17]</sup> presented a method for adding a parameter to a family of distributions with application to the exponential and Weibull families. Adamidis et al. <sup>[1]</sup> proposed the extended exponential-geometric (*EEG*) distribution which generalizes the *EG* distribution and discussed various of its statistical properties along with its reliability features. The hazard function of the *EEG* distribution can be monotone decreasing, increasing or constant. Barreto-Souza et al. <sup>[6]</sup> introduced a Weibull-geometric (*WG*) distribution following from compounded a Weibull distribution with a geometric distribution and obtained a new lifetime distribution with more

flexibility than the Weibull distribution and study some of its properties. Suppose that  $\{Y_i\}_{i=1}^Z$  are independent and identically distributed (*iid*) random variables following the Weibull distribution  $W(\alpha, \theta)$  with scale parameter  $\alpha > 0$ , shape parameter  $\theta > 0$  and probability density function

$$g(x; \alpha, \theta) = \theta \alpha^\theta x^{\theta-1} e^{-(\alpha x)^\theta}, x > 0,$$

and  $N$  a discrete random variable having a geometric distribution with probability function  $P(n; p) = (1-p)p^{n-1}$  for  $n \in N$  and  $p \in (0, 1)$ . Let  $X = \min \{Y_i\}_{i=1}^N$ . The marginal pdf of  $X$  is given by

$$f(x; p, \alpha, \theta) = \theta \alpha^\theta (1-p) x^{\theta-1} e^{-(\alpha x)^\theta} \left[1 - p e^{-(\alpha x)^\theta}\right]^{-2}, x > 0, \quad (1)$$

which defines the *WG* distribution (see Barreto-Souza et al. [6]). It is evident that (1) is much more flexible than the Weibull distribution. The *EG* distribution is a special case of the *WG* distribution for  $\theta = 1$ . When  $p$  approaches zero, the *WG* distribution leads to the Weibull  $W(\alpha, \theta)$  distribution. The corresponding cumulative distribution function and hazard rate function are given by

$$F(x; p, \alpha, \theta) = \frac{1 - e^{-(\alpha x)^\theta}}{1 - p e^{-(\alpha x)^\theta}}, x > 0, \quad (2)$$

and

$$h(x; p, \alpha, \theta) = \theta \alpha^\theta x^{\theta-1} \left[1 - p e^{-(\alpha x)^\theta}\right]^{-1}, \quad (3)$$

respectively.

In this article we defined the family of transmuted Weibull-geometric distribution. The main feature of this model is that a transmuted parameter is introduced in the subject distribution which provides greater flexibility in the form of new distributions. Using the quadratic rank transmutation map studied by Shaw et al. [18], we develop the four parameter transmuted Weibull-geometric  $TWG(x; p, \alpha, \theta, \lambda)$ . We provide a comprehensive description of mathematical properties of the subject distribution with the hope that it will attract wider applications in reliability, engineering and in other areas of research. The concept of transmuted explained in the following subsection.

## §2. Transmutation map

In this subsection we demonstrate transmuted probability distribution. Let  $F_1$  and  $F_2$  be the cumulative distribution functions, of two distributions with a common sample space. The general rank transmutation as given in Shaw et al. [18] is defined as

$$G_{R12}(u) = F_2(F_1^{-1}(u)) \text{ and } G_{R21}(u) = F_1(F_2^{-1}(u)).$$

Note that the inverse cumulative distribution function also known as quantile function is defined as

$$F^{-1}(y) = \inf_{x \in R} \{F(x) \geq y\} \text{ for } y \in [0, 1].$$

The functions  $G_{R12}(u)$  and  $G_{R21}(u)$  both map the unit interval  $I = [0, 1]$  into itself, and under suitable assumptions are mutual inverses and they satisfy  $G_{Rij}(0) = 0$  and  $G_{Rij}(1) = 1$ . A quadratic Rank Transmutation Map (QRTM) is defined as

$$G_{R12}(u) = u + \lambda u(1 - u), |\lambda| \leq 1, \quad (4)$$

from which it follows that the cdf's satisfy the relationship

$$F_2(x) = (1 + \lambda)F_1(x) - \lambda F_1(x)^2, \quad (5)$$

which on differentiation yields,

$$f_2(x) = f_1(x) [(1 + \lambda) - 2\lambda F_1(x)] \quad (6)$$

where  $f_1(x)$  and  $f_2(x)$  are the corresponding pdfs associated with cdf  $F_1(x)$  and  $F_2(x)$  respectively. An extensive information about the quadratic rank transmutation map is given in Shaw et al. [18] that at  $\lambda = 0$  we have the distribution of the base random variable.

Many authors dealing with the generalization of some well-known distributions. Aryal and Tsokos [4] defined the transmuted generalized extreme value distribution and they studied some basic mathematical characteristics of transmuted Gumbel probability distribution and it has been observed that the transmuted Gumbel can be used to model climate data. Also Aryal and Tsokos [3] presented a new generalization of Weibull distribution called the transmuted Weibull distribution. Recently, Aryal [18] proposed and studied the various structural properties of the transmuted Log-Logistic distribution, Muhammad Khan and King [10] introduced the transmuted modified Weibull distribution which extends recent development on transmuted Weibull distribution by Aryal et al. [3] and they studied the mathematical properties and maximum likelihood estimation of the unknown parameters. Merovci introduced the transmuted Rayleigh distribution, transmuted generalized Rayleigh distribution, and transmuted Lindley distribution and they studied the mathematical properties and maximum likelihood estimation of the unknown parameters. Elbatal [7] introduced transmuted modified inverse Weibull distribution.

### §3. Transmuted Weibull geometric distribution

In this section we studied the transmuted Weibull-geometric (TWG) distribution. Now using (1) and (2) we have the cdf of transmuted Weibull-geometric (TWG) distribution

$$F_{TWG}(x; p, \alpha, \theta, \lambda) = \frac{1 - e^{-(\alpha x)^\theta}}{1 - pe^{-(\alpha x)^\theta}} \left[ 1 + \lambda - \lambda \left( \frac{1 - e^{-(\alpha x)^\theta}}{1 - pe^{-(\alpha x)^\theta}} \right) \right], \quad (7)$$

where  $\lambda$  is the transmuted parameter. The corresponding probability density function (pdf) of the transmuted Weibull-geometric is given by

$$\begin{aligned} f_{TWG}(x; p, \alpha, \theta, \lambda) &= f_{WG}(x) [(1 + \lambda) - 2\lambda F_{WG}(x)] \\ &= \theta \alpha^\theta (1 - p) x^{\theta-1} e^{-(\alpha x)^\theta} [1 - pe^{-(\alpha x)^\theta}]^{-2} \\ &\quad \times \left\{ (1 + \lambda) - 2\lambda \left( \frac{1 - e^{-(\alpha x)^\theta}}{1 - pe^{-(\alpha x)^\theta}} \right) \right\}, \end{aligned} \quad (8)$$

respectively.

Figures (1) and (2) illustrates some of the possible shapes of the pdf and cdf of the TWG distribution for selected values of the parameters  $\theta, p, \alpha$  and  $\lambda$ , respectively.

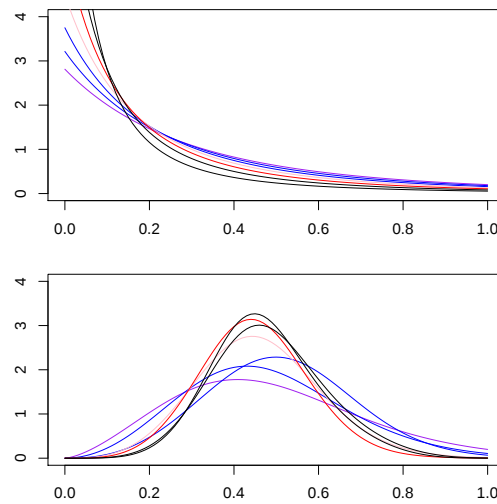


Figure 1: The pdf's of various TWG distributions for values of parameters:fig a)  $\theta = 1; \alpha = 1.5 \lambda = 0.5$ , fig b)  $\theta = 2.5, 3, 3.5, 4, 4.5, 5, 5.5; \lambda = 0.5, 0.6, 0.7, 0.8, 0.9, 0.4, 0.3$  and for  $p = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, \alpha = 1.5$ ; with color shapes purple, blue, plum, pink, red, black and darkcyan, respectively.

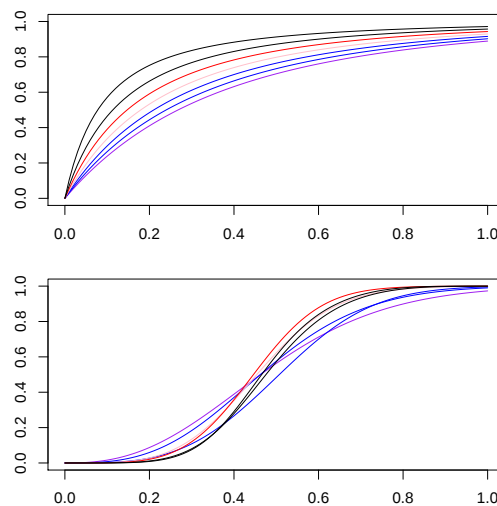


Figure 2: The cdf's of various TWG distributions for values of parameters:fig a)  $\theta = 1; \alpha = 1.5 \lambda = 0.5$ , fig b)  $\theta = 2.5, 3, 3.5, 4, 4.5, 5, 5.5; \lambda = 0.5, 0.6, 0.7, 0.8, 0.9, 0.4, 0.3$  and for  $p = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, \alpha = 1.5$ ; with color shapes purple, blue, plum, pink, red, black and darkcyan, respectively.

The reliability function ( $RF$ ) of the transmuted Weibull-geometric distribution is denoted by  $R_{TWG}(x)$  also known as the survivor function and is defined as

$$R_{TWG}(x) = 1 - F_{TWG}(x) = 1 - \frac{1 - e^{-(\alpha x)^\theta}}{1 - pe^{-(\alpha x)^\theta}} \left[ 1 + \lambda - \lambda \left( \frac{1 - e^{-(\alpha x)^\theta}}{1 - pe^{-(\alpha x)^\theta}} \right) \right]. \quad (9)$$

It is important to note that  $R_{TWG}(x) + F_{TWG}(x) = 1$ . One of the characteristic in reliability analysis is the hazard rate function ( $HF$ ) defined by

$$h_{TWG}(x) = \frac{f_{TWG}(x)}{1 - F_{TWG}(x)}. \quad (10)$$

It is important to note that the units for  $h_{TWG}(x)$  is the probability of failure per unit of time, distance or cycles. These failure rates are defined with different choices of parameters. The cumulative hazard function of the transmuted Weibull-geometric distribution is denoted by  $H_{TWG}(x)$  and is defined as

$$H_{TWG}(x) = -\ln \left| \frac{1 - e^{-(\alpha x)^\theta}}{1 - pe^{-(\alpha x)^\theta}} \left[ 1 + \lambda - \lambda \left( \frac{1 - e^{-(\alpha x)^\theta}}{1 - pe^{-(\alpha x)^\theta}} \right) \right] \right|. \quad (11)$$

It is important to note that the units for  $H_{TWG}(x)$  is the cumulative probability of failure per unit of time, distance or cycles. we can show that. For all choice of parameters the distribution has the decreasing patterns of cumulative instantaneous failure rates.

Figure 3 illustrates some of the possible shapes of the hazard function of the TWG distribution for selected values of the parameters  $\theta, p, \alpha$  and  $\lambda$ , respectively.

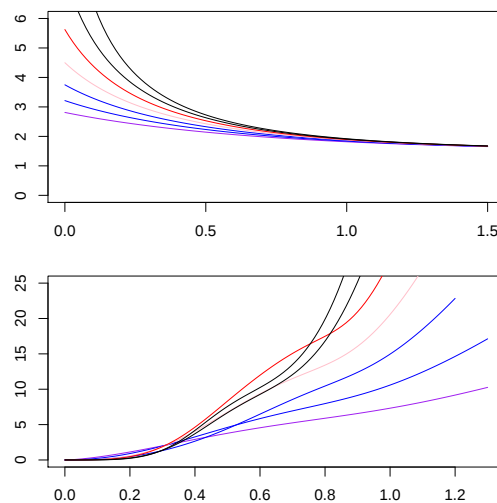


Figure 3: The hazard functions of various TWG distributions for values of parameters: fig a  $\theta = 1$ ,  $\alpha = 1.5$ ,  $\lambda = 0.5$ , fig b:  $\theta = 2.5, 3, 3.5, 4, 4.5, 5, 5.5$ ;  $\lambda = 0.5, 0.6, 0.7, 0.8, 0.9, 0.4, 0.3$  and for  $p = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$ ,  $\alpha = 1.5$ ; with color shapes purple, blue, plum, pink, red, black and darkcyan, respectively.

## §4. Statistical properties

This section is devoted to studying statistical properties of the (*TWG*) distribution.

### 4.1 Moments

In this subsection we discuss the  $r_{th}$  moment for (*TWG*) distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

**Theorem 3.1.** If  $X$  has  $TWG(x; \phi)$ ,  $\phi = (p, \alpha, \theta, \lambda)$  then the  $r_{th}$  moment of  $X$  is given by the following

$$\begin{aligned} \mu'_r(x) &= (1 + \lambda) \sum_{j=0}^{\infty} \frac{(j+1)(1-p)p^j \Gamma(\frac{r}{\theta} + 1)}{\alpha^r (j+1)^{\frac{r}{\theta}}} \\ &\quad - \lambda \sum_{j=0}^{\infty} \frac{(1-p)\Gamma(\frac{r}{\theta} + 1)}{\alpha^r} \left[ \frac{(j+2)}{(j+1)^{\frac{r}{\theta}}} - \frac{(j+1)}{(j+2)^{\frac{r}{\theta}}} \right]. \end{aligned} \quad (12)$$

**Proof.** Let  $X$  be a random variable with density function (8). The  $r_{th}$  ordinary moment of the (*TWG*) distribution is given by

$$\begin{aligned} \mu'_r(x) &= E(X^r) = \int_0^{\infty} x^r f(x, \Phi) dx \\ &= \theta \alpha^{\theta} (1-p) \left\{ (1+\lambda) \int_0^{\infty} x^{r+\theta-1} e^{-(\alpha x)^{\theta}} \left[ 1 - p e^{-(\alpha x)^{\theta}} \right]^{-2} \right. \\ &\quad \left. - 2\lambda \int_0^{\infty} x^{r+\theta-1} e^{-(\alpha x)^{\theta}} \left( 1 - e^{-(\alpha x)^{\theta}} \right) \left( 1 - p e^{-(\alpha x)^{\theta}} \right)^{-3} dx \right\}. \end{aligned} \quad (13)$$

If  $|z| < 1$  and  $k > 0$ , we have the series representation

$$(1-z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{\Gamma(k)j!} z^j. \quad (14)$$

Expanding the term  $(1 - p e^{-(\alpha x)^{\theta}})^{-3}$  as in (14) yields

$$\begin{aligned} \mu'_r(x) &= \theta \alpha^{\theta} (1-p) \left\{ (1+\lambda) \sum_{j=0}^{\infty} (j+1) p^j \int_0^{\infty} x^{r+\theta-1} e^{-(j+1)(\alpha x)^{\theta}} dx \right. \\ &\quad \left. - \lambda \sum_{j=0}^{\infty} (j+1)(j+2) p^j \int_0^{\infty} x^{r+\theta-1} e^{-(j+1)(\alpha x)^{\theta}} \left( 1 - e^{-(\alpha x)^{\theta}} \right) dx \right\} \\ &= (1+\lambda) \sum_{j=0}^{\infty} \frac{(j+1)(1-p)p^j \Gamma(\frac{r}{\theta} + 1)}{\alpha^r (j+1)^{\frac{r}{\theta}}} - \lambda \sum_{j=0}^{\infty} \frac{(1-p)\Gamma(\frac{r}{\theta} + 1)}{\alpha^r} \\ &\quad \times \left[ \frac{(j+2)}{(j+1)^{\frac{r}{\theta}}} - \frac{(j+1)}{(j+2)^{\frac{r}{\theta}}} \right]. \end{aligned} \quad (15)$$



Which completes the proof.

Based on the first four moments of the (*TWG*) distribution, the measures of skewness  $A(\Phi)$  and kurtosis  $k(\Phi)$  of the (*TWG*) distribution can be obtained as

$$A(\varphi) = \frac{\mu_3(\theta) - 3\mu_1(\theta)\mu_2(\theta) + 2\mu_1^3(\theta)}{[\mu_2(\theta) - \mu_1^2(\theta)]^{\frac{3}{2}}}$$

and

$$k(\varphi) = \frac{\mu_4(\theta) - 4\mu_1(\theta)\mu_3(\theta) + 6\mu_1^2(\theta)\mu_2(\theta) - 3\mu_1^4(\theta)}{[\mu_2(\theta) - \mu_1^2(\theta)]^2}.$$

#### 4.2 Moment Generating function

In this subsection we derived the moment generating function of (*TWG*) distribution.

**Theorem 3.2.** If  $X$  has (*TWG*) distribution, then the moment generating function  $M_X(t)$  has the following form

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left\{ (1+\lambda) \sum_{j=0}^{\infty} \frac{(j+1)(1-p)p^j \Gamma(\frac{r}{\theta} + 1)}{\alpha^r (j+1)^{\frac{r}{\theta}}} - \lambda \sum_{j=0}^{\infty} \frac{(1-p)\Gamma(\frac{r}{\theta} + 1)}{\alpha^r} \left[ \frac{(j+2)}{(j+1)^{\frac{r}{\theta}}} - \frac{(j+1)}{(j+2)^{\frac{r}{\theta}}} \right] \right\}. \quad (16)$$

**Proof.** We start with the well known definition of the moment generating function given by

$$\begin{aligned} M_X(t) &= \int_0^{\infty} e^{tx} f_{TWG}(x, \phi) dx \\ &= \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f_{TWG}(x, \phi) dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r' \\ &= \sum_{r=0}^{\infty} \frac{t^r}{r!} \left\{ (1+\lambda) \sum_{j=0}^{\infty} \frac{(j+1)(1-p)p^j \Gamma(\frac{r}{\theta} + 1)}{\alpha^r (j+1)^{\frac{r}{\theta}}} - \lambda \sum_{j=0}^{\infty} \frac{(1-p)\Gamma(\frac{r}{\theta} + 1)}{\alpha^r} \left[ \frac{(j+2)}{(j+1)^{\frac{r}{\theta}}} - \frac{(j+1)}{(j+2)^{\frac{r}{\theta}}} \right] \right\}. \end{aligned} \quad (17)$$

Which completes the proof.

## §5. Distribution of the order statistics

In this section, we derive closed form expressions for the pdfs of the  $r_{th}$  order statistic of the (*TWG*) distribution, also, the measures of skewness and kurtosis of the distribution of the  $r_{th}$  order statistic in a sample of size  $n$  for different choices of  $n; r$  are presented in this section. Let  $X_1, X_2, \dots, X_n$  be a simple random sample from (*TWG*) distribution with pdf and cdf given by (7) and (8), respectively. Let  $X_1, X_2, \dots, X_n$  denote the order statistics obtained from this sample. We now give the probability density function of  $X_{r:n}$ , say  $f_{r:n}(x, \Phi)$  and the moments

of  $X_{r:n}$ ,  $r = 1, 2, \dots, n$ . Therefore, the measures of skewness and kurtosis of the distribution of the  $X_{r:n}$  are presented. The probability density function of  $X_{r:n}$  is given by

$$f_{r:n}(x, \varphi) = \frac{1}{B(r, n-r+1)} [F(x, \varphi)]^{r-1} [1 - F(x, \varphi)]^{n-r} f(x, \varphi), \quad (18)$$

where  $F(x, \varphi)$  and  $f(x, \varphi)$  are the cdf and pdf of the (TWG) distribution given by (7), (8), respectively, and  $B(\cdot, \cdot)$  is the beta function. The pdf of the  $r_{th}$  order statistic for a transmuted weibull geometric distribution is given by

$$\begin{aligned} f_{r:n}(x, \varphi) &= \frac{1}{B(r, n-r+1)} \theta \alpha^\theta (1-p) x^{\theta-1} e^{-(\alpha x)^\theta} \left[1 - p e^{-(\alpha x)^\theta}\right]^{-2} \left\{ (1+\lambda) - 2\lambda \left( \frac{1 - e^{-(\alpha x)^\theta}}{1 - p e^{-(\alpha x)^\theta}} \right) \right\} \\ &\times \left\{ \frac{1 - e^{-(\alpha x)^\theta}}{1 - p e^{-(\alpha x)^\theta}} \left[ 1 + \lambda - \lambda \left( \frac{1 - e^{-(\alpha x)^\theta}}{1 - p e^{-(\alpha x)^\theta}} \right) \right] \right\}^{r-1} \\ &\times \left\{ 1 - \frac{1 - e^{-(\alpha x)^\theta}}{1 - p e^{-(\alpha x)^\theta}} \left[ 1 + \lambda - \lambda \left( \frac{1 - e^{-(\alpha x)^\theta}}{1 - p e^{-(\alpha x)^\theta}} \right) \right] \right\}^{n-r}. \end{aligned}$$

Therefore, and the pdf of the smallest order statistic  $X(1)$  is

$$\begin{aligned} f_{1:n}(x, \varphi) &= n \theta \alpha^\theta (1-p) x_{(1)}^{\theta-1} e^{-(\alpha x_{(1)})^\theta} \left[1 - p e^{-(\alpha x_{(1)})^\theta}\right]^{-2} \\ &\times \left\{ (1+\lambda) - 2\lambda \left( \frac{1 - e^{-(\alpha x_{(1)})^\theta}}{1 - p e^{-(\alpha x_{(1)})^\theta}} \right) \right\} \\ &\times \left\{ 1 - \frac{1 - e^{-(\alpha x_{(1)})^\theta}}{1 - p e^{-(\alpha x_{(1)})^\theta}} \left[ 1 + \lambda - \lambda \left( \frac{1 - e^{-(\alpha x_{(1)})^\theta}}{1 - p e^{-(\alpha x_{(1)})^\theta}} \right) \right] \right\}^{n-1}, \end{aligned}$$

and the pdf of the largest order statistic  $X(n)$  is given by

$$\begin{aligned} f_{n:n}(x, \varphi) &= n \theta \alpha^\theta (1-p) x_{(n)}^{\theta-1} e^{-(\alpha x_{(n)})^\theta} \left[1 - p e^{-(\alpha x_{(n)})^\theta}\right]^{-2} \\ &\times \left\{ (1+\lambda) - 2\lambda \left( \frac{1 - e^{-(\alpha x_{(n)})^\theta}}{1 - p e^{-(\alpha x_{(n)})^\theta}} \right) \right\} \\ &\times \left\{ \frac{1 - e^{-(\alpha x_{(n)})^\theta}}{1 - p e^{-(\alpha x_{(n)})^\theta}} \left[ 1 + \lambda - \lambda \left( \frac{1 - e^{-(\alpha x_{(n)})^\theta}}{1 - p e^{-(\alpha x_{(n)})^\theta}} \right) \right] \right\}^{n-1}. \end{aligned}$$

## §6. Least squares and Weighted least squares estimators

In this section we provide the regression based method estimators of the unknown parameters of the transmuted Weibull geometric distribution, which was originally suggested by Swain et al. [19] to estimate the parameters of beta distributions. It can be used some other cases also. Suppose  $Y_1, \dots, Y_n$  is a random sample of size  $n$  from a distribution function  $G(\cdot)$  and suppose  $Y_{(i)}$ ;  $i = 1, 2, \dots, n$  denotes the ordered sample. The proposed method uses the distribution of

$G(Y_{(i)})$ . For a sample of size  $n$ , we have

$$E(G(Y_{(j)})) = \frac{j}{n+1}, V(G(Y_{(j)})) = \frac{j(n-j+1)}{(n+1)^2(n+2)}$$

and  $Cov(G(Y_{(j)}), G(Y_{(k)})) = \frac{j(n-k+1)}{(n+1)^2(n+2)}$ ; for  $j < k$ ,

see Johnson et al. [8]. Using the expectations and the variances, two variants of the least squares methods can be used.

**Method 1. (Least Squares Estimators)** Obtain the estimators by minimizing

$$\sum_{j=1}^n \left( G(Y_{(j)}) - \frac{j}{n+1} \right)^2, \quad (6.1)$$

with respect to the unknown parameters. Therefore in case of  $TWG$  distribution the least squares estimators of  $p, \alpha, \theta$ , and  $\lambda$ , say,  $\hat{p}_{LSE}, \hat{\alpha}_{LSE}, \hat{\theta}_{LSE}$  and  $\hat{\lambda}_{LSE}$  respectively, can be obtained by minimizing

$$\sum_{j=1}^n \left[ \frac{1 - e^{-(\alpha x_j)^\theta}}{1 - pe^{-(\alpha x_j)^\theta}} \left[ 1 + \lambda - \lambda \left( \frac{1 - e^{-(\alpha x_j)^\theta}}{1 - pe^{-(\alpha x_j)^\theta}} \right) \right] - \frac{j}{n+1} \right]^2$$

with respect to  $\theta, p$  and  $\lambda$ .

**Method 2. (Weighted Least Squares Estimators)** The weighted least squares estimators can be obtained by minimizing

$$\sum_{j=1}^n w_j \left( G(Y_{(j)}) - \frac{j}{n+1} \right)^2, \quad (19)$$

with respect to the unknown parameters, where

$$w_j = \frac{1}{V(G(Y_{(j)}))} = \frac{(n+1)^2(n+2)}{j(n-j+1)}.$$

Therefore, in case of  $TWG$  distribution the weighted least squares estimators of  $p, \alpha, \theta$  and  $\lambda$ , say,  $\hat{p}_{WLSE}, \hat{\alpha}_{WLSE}, \hat{\theta}_{WLSE}$  and  $\hat{\lambda}_{WLSE}$  respectively, can be obtained by minimizing

$$\sum_{j=1}^n w_j \left[ \frac{1 - e^{-(\alpha x_j)^\theta}}{1 - pe^{-(\alpha x_j)^\theta}} \left[ 1 + \lambda - \lambda \left( \frac{1 - e^{-(\alpha x_j)^\theta}}{1 - pe^{-(\alpha x_j)^\theta}} \right) \right] - \frac{j}{n+1} \right]^2,$$

with respect to the unknown parameters only.

## §7. Estimation and inference

In this section, we determine the maximum likelihood estimates ( $MLEs$ ) of the parameters of the ( $TWG$ ) distribution from complete samples only. Let  $X_1, X_2, \dots, X_n$  be a random sample of size  $n$  from  $TWG(x; \phi), \phi = (p, \alpha, \theta, \lambda)$ . The likelihood function for the vector of parameters

can be written as

$$\begin{aligned}
 Lf(x_{(i)}, \phi) &= \prod_{i=1}^n f(x_{(i)}, \phi) \\
 &= (\theta \alpha^\theta (1-p))^n \prod_{i=1}^n (x_i)^{\theta-1} e^{-\sum_{i=1}^n (\alpha x_i)^\theta} \prod_{i=1}^n [1 - pe^{-(\alpha x_i)^\theta}]^{-2} \\
 &\quad \times \prod_{i=1}^n \left\{ (1+\lambda) - 2\lambda \left( \frac{1 - e^{-(\alpha x_i)^\theta}}{1 - pe^{-(\alpha x_i)^\theta}} \right) \right\}. \quad (20)
 \end{aligned}$$

Taking the log-likelihood function for the vector of parameters  $\phi = (p, \alpha, \theta, \lambda)$  we get

$$\begin{aligned}
 \log L &= n \log \theta + n \log \alpha + n \log(1-p) + (\theta-1) \sum_{i=1}^n \log(x_i) - \theta \sum_{i=1}^n (\alpha x_i) - 2 \sum_{i=1}^n \log [1 - pe^{-(\alpha x_i)^\theta}] \\
 &\quad + \sum_{i=1}^n \log \left\{ (1+\lambda) - 2\lambda \left( \frac{1 - e^{-(\alpha x_i)^\theta}}{1 - pe^{-(\alpha x_i)^\theta}} \right) \right\}. \quad (21)
 \end{aligned}$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (21). The components of the score vector are given by

$$\begin{aligned}
 \frac{\partial \log L}{\partial \theta} &= \frac{n}{\theta} + n \log \alpha + \sum_{i=1}^n \log(x_i) - \sum_{i=1}^n (\alpha x_i) - 2 \sum_{i=1}^n \frac{pe^{-(\alpha x_i)^\theta} (\alpha x_i)^\theta \log(\alpha x_i)}{[1 - pe^{-(\alpha x_i)^\theta}]} \\
 &\quad + \sum_{i=1}^n \frac{1}{\left\{ (1+\lambda) - 2\lambda \left( \frac{1 - e^{-(\alpha x_i)^\theta}}{1 - pe^{-(\alpha x_i)^\theta}} \right) \right\}} \left[ \frac{(1-p)e^{-(\alpha x_i)^\theta} (\alpha x_i)^\theta \log(\alpha x_i)}{[1 - pe^{-(\alpha x_i)^\theta}]^2} \right] \\
 &= 0, \quad (22)
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial \log L}{\partial \alpha} &= \frac{n\theta}{\alpha} - \theta \sum_{i=1}^n (x_i) - 2 \sum_{i=1}^n \frac{\theta pe^{-(\alpha x_i)^\theta} x_i (\alpha x_i)^{\theta-1}}{[1 - pe^{-(\alpha x_i)^\theta}]} \\
 &\quad - 2\lambda \sum_{i=1}^n \frac{1}{\left\{ (1+\lambda) - 2\lambda \left( \frac{1 - e^{-(\alpha x_i)^\theta}}{1 - pe^{-(\alpha x_i)^\theta}} \right) \right\}} \left[ \frac{(1-p)\theta e^{-(\alpha x_i)^\theta} x_i (\alpha x_i)^{\theta-1}}{[1 - pe^{-(\alpha x_i)^\theta}]^2} \right] = 0, \quad (23)
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial \log L}{\partial p} &= \frac{-n}{1-p} + 2 \sum_{i=1}^n \frac{e^{-(\alpha x_i)^\theta}}{[1 - pe^{-(\alpha x_i)^\theta}]} \\
 &\quad - 2\lambda \sum_{i=1}^n \frac{1}{\left\{ (1+\lambda) - 2\lambda \left( \frac{1 - e^{-(\alpha x_i)^\theta}}{1 - pe^{-(\alpha x_i)^\theta}} \right) \right\}} \left[ \frac{e^{-(\alpha x_i)^\theta} (1 - e^{-(\alpha x_i)^\theta})}{[1 - pe^{-(\alpha x_i)^\theta}]^2} \right] = 0, \quad (24)
 \end{aligned}$$

and

$$\frac{\partial \log L}{\partial \lambda} = \sum_{i=1}^n \frac{1 - 2 \left( \frac{1 - e^{-(\alpha x_i)^\theta}}{1 - pe^{-(\alpha x_i)^\theta}} \right)}{\left\{ (1+\lambda) - 2\lambda \left( \frac{1 - e^{-(\alpha x_i)^\theta}}{1 - pe^{-(\alpha x_i)^\theta}} \right) \right\}} = 0. \quad (25)$$

We can find the estimates of the unknown parameters by maximum likelihood method by setting these above non-linear equations (22)-(25) to zero and solve them simultaneously. Therefore, we have to use mathematical package to get the *MLE* of the unknown parameters.

|     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 194 | 413 | 90  | 74  | 55  | 23  | 97  | 50  | 359 | 50  | 130 | 487 | 102 | 15  | 14  | 10  | 57  |
| 320 | 261 | 51  | 44  | 9   | 254 | 493 | 18  | 209 | 41  | 58  | 60  | 48  | 56  | 87  | 11  | 102 |
| 12  | 5   | 100 | 14  | 29  | 37  | 186 | 29  | 104 | 7   | 4   | 72  | 270 | 283 | 7   | 57  | 33  |
| 100 | 61  | 502 | 220 | 120 | 141 | 22  | 603 | 35  | 98  | 54  | 181 | 65  | 49  | 12  | 239 | 14  |
| 18  | 39  | 3   | 12  | 5   | 32  | 9   | 14  | 70  | 47  | 62  | 142 | 3   | 104 | 85  | 67  | 169 |
| 24  | 21  | 246 | 47  | 68  | 15  | 2   | 91  | 59  | 447 | 56  | 29  | 176 | 225 | 77  | 197 | 438 |
| 43  | 134 | 184 | 20  | 386 | 182 | 71  | 80  | 188 | 230 | 152 | 36  | 79  | 59  | 33  | 246 | 1   |
| 79  | 3   | 27  | 201 | 84  | 27  | 21  | 16  | 88  | 130 | 14  | 118 | 44  | 15  | 42  | 106 | 46  |
| 230 | 59  | 153 | 104 | 20  | 206 | 5   | 66  | 34  | 29  | 26  | 35  | 5   | 82  | 5   | 61  | 31  |
| 118 | 326 | 12  | 54  | 36  | 34  | 18  | 25  | 120 | 31  | 22  | 18  | 156 | 11  | 216 | 139 | 67  |
| 310 | 3   | 46  | 210 | 57  | 76  | 14  | 111 | 97  | 62  | 26  | 71  | 39  | 30  | 7   | 44  | 11  |
| 63  | 23  | 22  | 23  | 14  | 18  | 13  | 34  | 62  | 11  | 191 | 14  | 16  | 18  | 130 | 90  | 163 |
| 208 | 1   | 24  | 70  | 16  | 101 | 52  | 208 | 95  |     |     |     |     |     |     |     |     |

## §8. Application

The first data set given in table 1 consist of the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes. The pooled data with 213 observations were first analyzed by Proschan <sup>[20]</sup> and discussed further by Adamidis et al. <sup>[20]</sup> and Kus <sup>[9]</sup>.

In order to compare the two distribution models, we consider criteria like  $-2\ell$ , AIC (Akaike information criterion) and AICC (corrected Akaike information criterion) for the data set. The better distribution corresponds to smaller  $-2\ell$ , AIC and AICC values:

$$AIC = 2k - 2\ell \quad \text{and} \quad AICC = AIC + \frac{2k(k+1)}{n-k-1},$$

where  $k$  is the number of parameters in the statistical model,  $n$  the sample size and  $\ell$  is the maximized value of the log-likelihood function under the considered model.

Table 1 shows parameters  $MLEs$ ,  $-\log(L)$ ,  $AIC$  and  $K-S$  values to each one of the two fitted distributions for data set. The values in Tables 1, indicate that the  $TWG$  is a strong competitor to other distribution used here for fitting data set. A density plot compares the fitted densities of the models with the empirical histogram of the observed data (Fig. fig4 ). The fitted density for the  $TWG$  model is closer to the empirical histogram than the fits of the  $WG$  model.

Table 1: The ML estimates, Log-likelihood, AIC and K-S for data set

| Model | ML Estimates                         | -LL      | K-S    | AIC      |
|-------|--------------------------------------|----------|--------|----------|
| TWG   | $\hat{\theta} = 1.027$               | 1178.264 | 0.0343 | 2364.528 |
|       | $\hat{p} = 0.844$                    |          |        |          |
|       | $\hat{\alpha} = 0.545 \cdot 10^{-2}$ |          |        |          |
|       | $\hat{\lambda} = -0.762$             |          |        |          |
| WG    | $\hat{\theta} = 1.337$               | 1181.947 | 0.0398 | 2369.894 |
|       | $\hat{p} = 0.999$                    |          |        |          |
|       | $\hat{\alpha} = 1.729 \cdot 10^{-5}$ |          |        |          |

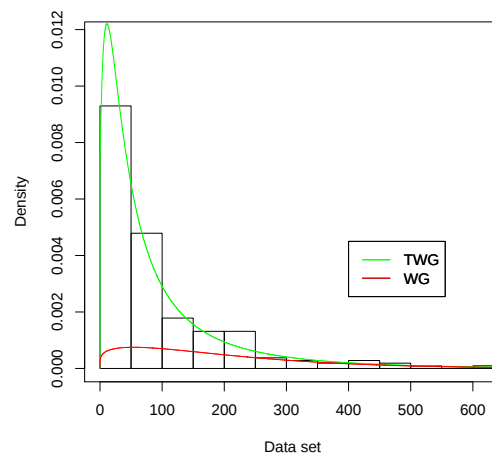


Figure 4: Estimated densities of the models for data set.

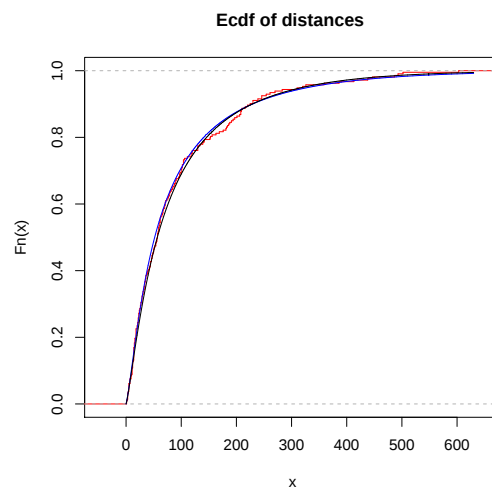


Figure 5: Empirical, cdf for TWG and WG for data set .

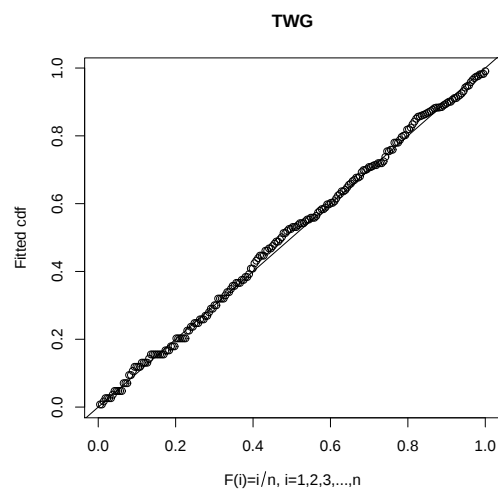


Figure 6: PP plots for fitted TWG for data set.

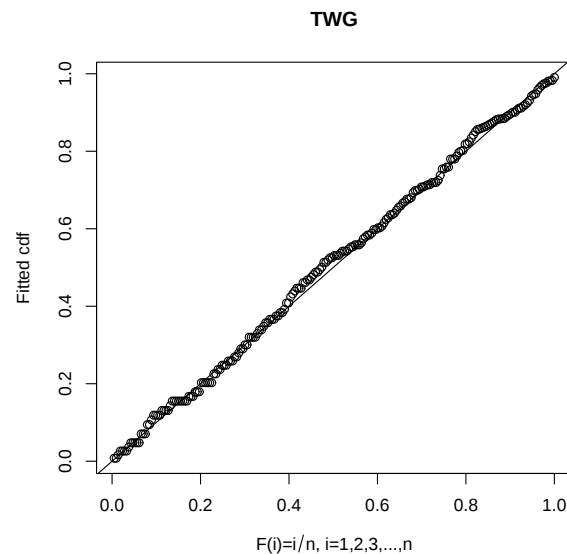


Figure 7: PP plots for fitted WG for data set.

## References

- [1] K. Adamidis, T. Dimitrakopoulou and S. Loukas, On a generalization of the exponential-geometric distribution, *Statist. and Probab. Lett.*, **73**(2005), 259-269.
- [2] K. Adamidis and S. Loukas, A lifetime distribution with decreasing failure rate, *Statist. and Probab. Lett.*, **39**(1998), 35-42.
- [3] G. R. Aryall and C. P. Tsokos, Transmuted Weibull distribution: a generalization of the Weibull probability distribution, *European Journal of Pure and Applied Mathematics*, **4**(2011), No. 2, 89-102.
- [4] G. R. Aryall and C. P. Tsokos, On the transmuted extreme value distribution with applications, *Nonlinear Analysis: Theory, Methods and Applications*, **71**(2009), 1401-1407.
- [5] G. R. Aryall, Transmuted Log-Logistic Distribution, *J. Stat. Appl. Pro.* **2**, (2013), No. 1, 11-20.
- [6] Barreto-Souza, Alice Lemos de Moraes and Gauss M. Cordeiro, *The Weibull-geometric distribution*, 2008.
- [7] I. Elbatal, Transmuted modified inverse Weibull distribution, *International Journal of Mathematical Archive*, **4**(2013), No. 8, 117-129.
- [8] N. L. Johnson, S. Kotz, and N. Balakrishnan, *Continuous univariate distributions*, Wiley, New York, **2**(1995).
- [9] C. A. Kus, New lifetime distribution, *Computat. Statist. Data Analysis*, **51**(2007), 449-450.
- [10] Muhammad Shuaib Khan and Robert King, Transmuted modified Weibull distribution: a generalization of the modified Weibull probability distribution *European Journal of Pure and*



Applied Mathematics, **6**( 2013), No. 1, 66-88.

[11] F. Merovci, Transmuted rayleigh distribution, Austrian Journal of Statistics, **42**(2013), No. 1, 21-31.

[12] F. Merovci, Transmuted generalized Rayleigh distribution. Journal of Statistics Applications and Probability, **3**(2014), No. 1, 9-20.

[13] F. Merovci, Transmuted Lindley distribution, International Journal of Open Problems in Computer Science and Mathematics, **6**(2013), No. 2, 63-72.

[14] F. Merovci and L. Puka, Transmuted Pareto distribution, ProbStat Forum, **7**(2014), 1-11.

[15] F. Merovci and I. Elbatal, Transmuted Lindley-geometric distribution and its applications, Journal of Statistics Applications and Probability, **3**(2014), No. 1, 77-91.

[16] F. Merovci, I. Elbatal and Ahmed Alaa, The transmuted generalized inverse Weibull distribution, Austrian Journal of Statistics, **43**(2014), No. 2, 29-41.

[17] AW. Marshall and I. Olkin, A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families, Biometrika, **84**(1997), 641-652.

[18] W. Shaw and I. Buckley, The alchemy of probability distributions: beyond Gram-Charlier expansions and a skew- kurtotic- normal distribution from a rank transmutation map, 2007.

[19] J. J. Swain, S. Venkatraman and J. R. Wilson, Least-squares estimation of distribution functions in Johnson's translation system. Journal of Statistical Computation and Simulation, **29**(1988), No. 4, 271-297.

[20] F. Proschan, Theoretical explanation of observed decreasing failure rate, Technometrics, (1963), No. 5, 375-383.

# A sheaf construction on the primary-like spectrum of modules

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**Abstract** Let  $M$  be a unitary module over a commutative ring  $R$  with identity. Let  $\text{Spec}_L(M)$  denote the primary-like spectrum of  $M$  which has been equipped with the Zariski topology  $\mathcal{T}$ . In this paper, we obtain an  $R$ -module  $\mathcal{O}_{\text{Spec}_L(M)}(\mathcal{U})$  for each open set  $\mathcal{U}$  of the topological space  $(\text{Spec}_L(M), \mathcal{T})$ . We show that  $\mathcal{O}_{\text{Spec}_L(M)}$  is a sheaf of  $R$ -modules over  $\text{Spec}_L(M)$ .

**Keywords** Primary-like submodule, Zariski topology, Sheaf of a module.

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## §1. Introduction and preliminaries

All rings in this article are commutative with identity and modules are unitary. For a submodule  $N$  of an  $R$ -module  $M$  we denote the annihilator of the factor module  $M/N$  by  $(N : M)$ , i.e.,  $(N : M) = \{r \in R \mid rM \subseteq N\}$ . We call  $M$  faithful if  $\text{Ann}(M) = (0 : M) = 0$ .

A submodule  $P$  of an  $R$ -module  $M$  is said to be  $p$ -prime if  $P \neq M$  and for  $p = (P : M)$ , whenever  $rm \in P$  (where  $r \in R$  and  $m \in M$ ), then  $m \in P$  or  $r \in p$ . The collection of all prime submodules of  $M$  is denoted by  $\text{Spec}(M)$ .<sup>[8]</sup> For a submodule  $N$  of  $M$  the radical of  $N$ , denoted by  $\text{rad}N$ , is the intersection of all prime submodules of  $M$  containing  $N$ , unless no such primes exist, in which case  $\text{rad}N = M$ .<sup>[9]</sup> For an ideal  $I$  of  $R$ , the radical of  $I$  is denoted by  $\sqrt{I}$ .

A proper submodule  $Q$  of  $M$  is primary-like provided that  $rm \in Q$ , for  $r \in R$  and  $m \in M$ , implies  $r \in (Q : M)$  or  $x \in \text{rad}Q$ .<sup>[2]</sup> If  $\sqrt{(Q : M)} = p$  is a prime ideal, then  $Q$  is also called a  $p$ -primary-like submodule of  $M$ .<sup>[2]</sup>

An  $R$ -module  $M$  is said to be primeful if either  $M = 0$  or for every  $p \in V(\text{Ann}(M))$ , there exists  $P \in \text{Spec}(M)$  such that  $(P : M) = p$ .<sup>[5]</sup> If  $M/N$  is a primeful  $R$ -module, we say that  $N$  satisfies the primeful property. In this case  $(\text{rad}N : M) = \sqrt{(N : M)}$ .<sup>[5]</sup> It is easily seen that if  $Q$  is a primary-like submodule of the  $R$ -module  $M$  satisfying the primeful property, then  $(Q : M)$  is a primary ideal of  $R$  and so  $Q$  is a  $p$ -primary-like submodule of  $M$ , where  $p = \sqrt{(Q : M)} = (\text{rad}Q : M)$ .

The primary-like spectrum  $\text{Spec}_L(M)$  is defined to be the set of all primary-like submodules of  $M$  satisfying the primeful property.<sup>[2]</sup>  $\text{Spec}_L(M)$  may be empty for some non-zero modules  $M$ . For example  $\text{Spec}_L(\mathbb{Q}) = \emptyset$  while  $\text{Spec}(\mathbb{Q}) = \{0\}$ , where  $\mathbb{Q}$  is the module of rational numbers over the ring of integers  $\mathbb{Z}$ . Throughout the rest of this paper, we assume that  $\text{Spec}_L(M)$  is

non-empty.

In the literature, there are many different generalizations of the Zariski topology for modules over commutative rings. For example, Lu defined a Zariski topology on  $\text{Spec}(M)$  whose closed sets are  $V(N) = \{P \in \text{Spec}(M) \mid (N : M) \subseteq (P : M)\}$  for any submodule  $N$  of  $M$ .<sup>[7]</sup> As a new generalization of the Zariski topology, we introduce the Zariski topology on  $\text{Spec}_L(M)$  for any  $R$ -module  $M$  in which closed sets are varieties  $\nu(N) = \{Q \in \text{Spec}_L(M) \mid \sqrt{(N : M)} \subseteq \sqrt{(Q : M)}\}$  of all submodules  $N$  of  $M$ . In the following lemma we show that, if  $\eta(M)$  denotes the collection of all subsets  $\nu(N)$  of  $\text{Spec}_L(M)$ , then  $\eta(M)$  satisfies the axioms for the closed subsets of a topological space on  $\text{Spec}_L(M)$ , called Zariski topology and denoted by  $\mathcal{T}$ .

**Lemma 1.1.** Let  $M$  be an  $R$ -module. Then for submodules  $N$ ,  $N'$  and  $\{N_i \mid i \in I\}$  of  $M$  we have

$$(1) \nu(0) = \text{Spec}_L(M) \text{ and } \nu(M) = \emptyset.$$

$$(2) \bigcap_{i \in I} \nu(N_i) = \nu\left(\sum_{i \in I} (N_i : M)M\right).$$

$$(3) \nu(N) \cup \nu(N') = \nu(N \cap N').$$

**Proof.** (1) and (3) are trivial. (2) follows from the following implications:

$$\begin{aligned} Q \in \bigcap_{i \in I} \nu(N_i) &\Rightarrow \sqrt{(Q : M)} \supseteq \sqrt{(N_i : M)} \quad \forall i \in I \\ &\Rightarrow \sqrt{(Q : M)} \supseteq (N_i : M) \quad \forall i \in I \\ &\Rightarrow \sqrt{(Q : M)} \supseteq \sum_{i \in I} (N_i : M) \\ &\Rightarrow \sqrt{(Q : M)}M \supseteq \sum_{i \in I} (N_i : M)M \\ &\Rightarrow (\sqrt{(Q : M)}M : M) \supseteq \left(\sum_{i \in I} (N_i : M)M : M\right) \\ &\Rightarrow ((\text{rad } Q : M)M : M) \supseteq \left(\sum_{i \in I} (N_i : M)M : M\right) \\ &\Rightarrow (\text{rad } Q : M) \supseteq \left(\sum_{i \in I} (N_i : M)M : M\right) \\ &\Rightarrow \sqrt{(Q : M)} \supseteq \sqrt{\left(\sum_{i \in I} (N_i : M)M : M\right)} \\ &\Rightarrow Q \in \nu\left(\sum_{i \in I} (N_i : M)M\right). \end{aligned}$$

For the reverse inclusion we have

$$\begin{aligned}
Q \in \nu\left(\sum_{i \in I} (N_i : M)M\right) &\Rightarrow \sqrt{(Q : M)} \supseteq \sqrt{\left(\sum_{i \in I} (N_i : M)M : M\right)} \\
&\Rightarrow \sqrt{(Q : M)} \supseteq \left(\sum_{i \in I} (N_i : M)M : M\right) \\
&\Rightarrow \sqrt{(Q : M)} \supseteq ((N_i : M)M : M) \quad \forall i \in I \\
&\Rightarrow \sqrt{(Q : M)} \supseteq (N_i : M) \quad \forall i \in I \\
&\Rightarrow \sqrt{(Q : M)} \supseteq \sqrt{(N_i : M)} \quad \forall i \in I \\
&\Rightarrow Q \in \bigcap_{i \in I} \nu(N_i)
\end{aligned}$$

It is well-known that for any commutative ring  $R$ , there is a sheaf of rings on  $\text{Spec}(R)$ , called the structure sheaf, denoted by  $\mathcal{O}_{\text{Spec}(R)}$ , defined as follows: for each prime ideal  $p$  of  $R$ , let  $R_p$  be the localization of  $R$  at  $p$ . For an open set  $U$  of  $\text{Spec}(R)$ , we define  $\mathcal{O}_{\text{Spec}(R)}(U)$  to be the set of functions  $s : U \rightarrow \prod_{p \in U} R_p$ , such that  $s(p) \in R_p$ , for each  $p \in U$ , and such that  $s$  is locally a quotient of elements of  $R$ : to be precise, we require that for each  $p \in U$ , there is a neighborhood  $V$  of  $p$ , contained in  $U$ , and elements  $a, f \in R$ , such that for each  $q \in V$ ,  $f \notin q$ , and  $s(q) = \frac{a}{f}$  in  $R_q$  (see for example [3], for definition and basic properties of the sheaf  $\mathcal{O}_{\text{Spec}(R)}$ ).

Let  $M$  be a primeful faithful  $R$ -module. In [10], the author obtained an  $R$ -module  $\mathcal{O}_{\text{Spec}(M)}(U)$  for each open set  $U$  in  $\text{Spec}(M)$  (with respect to the Zariski topology) such that  $\mathcal{O}_{\text{Spec}(M)}$  is a sheaf of modules over  $\text{Spec}(M)$  which is a generalization of the structure sheaf of rings. As a new generalization, we obtain an  $R$ -module  $\mathcal{O}_{\text{Spec}_L(M)}(\mathcal{U})$  for any open set  $\mathcal{U}$  of  $\text{Spec}_L(M)$  (with respect to the Zariski topology) to construct a sheaf  $\mathcal{O}_{\text{Spec}_L(M)}$  of  $R$ -modules over  $\text{Spec}_L(M)$  (Theorem 2.5).

## §2. A sheaf of modules

For each  $P \in \text{Spec}(M)$ , we put  $\nu_P = \{X_r \mid P \in X_r\}$ , where  $X_r = \text{Spec}(M) - V(rM)$ . Also for each  $r \in R$  we consider the  $R$ -module of fractions  $M_r = \{\frac{m}{r^n} \mid m \in M, n \in \mathbb{N}\}$ . In [10], it is proved that if  $M$  is a faithful primeful  $R$ -module and  $X_s \subseteq X_r$ , then there exists an  $R$ -module homomorphism  $\rho_{X_s, X_r}^* : M_r \rightarrow M_s$  defined by  $\frac{m}{r^n} \mapsto \frac{a^n m}{s^{nk}}$  for some positive integer  $k$  and  $a \in R$ . Moreover the inductive limit  $\lim_{X_r \in \nu_P} M_r$  of the direct system  $\{M_r, \rho_{X_s, X_r}^*\}$  has been introduced as a  $\lim_{D_r \in \nu_p} R_r$ -module where  $(P : M) = p$  and  $\nu_p = \{D_r \mid p \in D_r\}$ . In particular, it has been shown that if  $M$  is a faithful primeful  $R$ -module, then  $\lim_{X_r \in \nu_P} M_r \cong M_p$ , where  $p = (P : M)$ .<sup>[10]</sup> In this section, we obtain an inductive limit of a new direct system similarly. In particular, we investigate sufficient conditions under which these limits are isomorphic as  $R_p$ -modules, where  $p = \sqrt{(Q : M)}$  for some  $Q \in \text{Spec}_L(M)$  (Corollary 2.2). We will use  $\bar{R}$ ,  $X^{\bar{R}}$  and  $\mathcal{X}$  to represent  $R/\text{Ann}(M)$ ,  $\text{Spec}(R/\text{Ann}(M))$  and  $\text{Spec}_L(M)$ , respectively.

**Lemma 2.1.** Let  $r$  be a non-nilpotent element of a ring  $R$ ,  $S$  be the multiplicatively closed subset  $\{1, r, r^2, r^3, \dots, r^t, \dots\}$  and  $M$  be an  $R$ -module. Let  $Q \in \mathcal{X}$ . Then  $\sqrt{(rM : M)} \not\subseteq \sqrt{(Q : M)}$  if and only if  $\sqrt{(Q : M)} \cap S = \emptyset$ .

**Proof.** Suppose that  $\sqrt{(rM : M)} \not\subseteq \sqrt{(Q : M)}$ . Therefore  $r \notin \sqrt{(Q : M)}$ . Since  $\sqrt{(Q : M)}$  is a prime ideal of  $R$ , then  $1, r^2, r^3, \dots \notin \sqrt{(Q : M)}$ . Thus  $\sqrt{(Q : M)} \cap S = \emptyset$ . Conversely, assume  $\sqrt{(Q : M)} \cap S = \emptyset$  and  $\sqrt{(rM : M)} \subseteq \sqrt{(Q : M)}$ . Hence  $r \in \sqrt{(Q : M)} \cap S$ , a contradiction. Thus  $\sqrt{(rM : M)} \not\subseteq \sqrt{(Q : M)}$ .

For each  $r \in R$ , we set  $\mathcal{X}_r = \mathcal{X} - \nu(rM)$  and  $D_{\bar{r}} = X^{\bar{R}} - V(\bar{R}\bar{r})$ . It is easily seen that  $\mathcal{X}_{0_R} = \emptyset$ ,  $\mathcal{X}_{1_R} = \mathcal{X}$ .

**Lemma 2.2.** Let  $r$  be a non-nilpotent element of a ring  $R$  and  $M$  an  $R$ -module. Then  $\mathcal{X}_r = \{Q \in \mathcal{X} \mid r \notin \sqrt{(Q : M)}\}$ .

**Proof.**

$$\begin{aligned} \nu(rM) &= \{Q \in \mathcal{X} \mid \sqrt{(Q : M)} \supseteq \sqrt{(rM : M)}\} \\ &= \{Q \in \mathcal{X} \mid \sqrt{(Q : M)} \supseteq (rM : M)\} \\ &= \{Q \in \mathcal{X} \mid \sqrt{(Q : M)} \cap S \neq \emptyset\}, \text{ by Lemma 2.1} \\ &= \{Q \in \mathcal{X} \mid r \in \sqrt{(Q : M)}\}. \end{aligned}$$

Thus  $\mathcal{X}_r = \mathcal{X} - \nu(rM) = \{Q \in \mathcal{X} \mid r \notin \sqrt{(Q : M)}\}$ .

**Theorem 2.1.** Let  $M$  be an  $R$ -module and  $Q \in \mathcal{X}$ . Let  $S$  be a multiplicatively closed subset of  $R$  such that  $S \cap \sqrt{(Q : M)} = \emptyset$ . Then  $S^{-1}Q$  is a primary-like submodule of  $S^{-1}R$ -module  $S^{-1}M$  satisfying the primeful property.

**Proof.** It is easy to see that  $m/1 \in S^{-1}M \setminus S^{-1}Q$  for each  $m \in M \setminus Q$  and so  $S^{-1}Q \neq S^{-1}M$ . Suppose  $(r/s)(x/t) \in S^{-1}Q$  and  $r/s \notin (S^{-1}Q : S^{-1}M)$  for  $r/s \in S^{-1}R$  and  $x/t \in S^{-1}M$ . Since  $S^{-1}((Q : M)) \subseteq (S^{-1}Q : S^{-1}M)$ , then  $r \notin (Q : M)$ . Thus there exist  $u, w \in S$ ,  $q \in Q$  such that  $wurx = wstq$ . It follows that  $x \in \text{rad } Q$  since  $Q$  is primary-like. Thus  $x/t \in S^{-1}(\text{rad } Q) \subseteq \text{rad } (S^{-1}Q)$ , by [9]. Thus  $S^{-1}Q$  is a primary-like submodule of  $S^{-1}M$ . Let  $S^{-1}p$  be a prime ideal of  $S^{-1}R$  containing  $(S^{-1}Q : S^{-1}M)$ . Hence  $p \cap S = \emptyset$  and,

$$\begin{aligned} S^{-1}((Q : M)) &\subseteq (S^{-1}Q : S^{-1}M) \subseteq S^{-1}p \\ &\Rightarrow S^{-1}(\sqrt{(Q : M)}) \subseteq S^{-1}p \\ &\Rightarrow \sqrt{(Q : M)} \subseteq p. \end{aligned}$$

Since  $Q$  satisfies the primeful property, there exists a  $p$ -prime submodule  $P$  of  $M$  containing  $Q$ . By [8], since  $p \cap S = \emptyset$ ,  $S^{-1}P$  is an  $S^{-1}p$ -prime submodule of  $S^{-1}M$  containing  $S^{-1}Q$ , i.e.,  $S^{-1}Q$  satisfies the primeful property.

Combining Lemma 2.2 and Theorem 2.1, we can easily deduce the following:

**Corollary 2.1.** Let  $r$  be a non-nilpotent element of  $R$  and  $S$  be the multiplicatively closed subset  $\{1, r, r^2, r^3, \dots\}$ . If  $Q \in \nu(rM)$ , then  $S^{-1}Q \in \text{Spec}_L(S^{-1}M)$ .

For any  $R$ -module  $M$ , we define the map  $\phi : \mathcal{X} \rightarrow X^{\bar{R}}$  given by  $\phi(Q) = \overline{\sqrt{(Q : M)}}$ . The following results offers a useful piece of information for the map  $\phi$ .

**Proposition 2.1.** Let  $M$  be an  $R$ -module. Then  $\phi^{-1}(V^{\bar{R}}(\bar{I})) = \nu(IM)$ , for every ideal  $I \in V(\text{Ann}(M))$ . Therefore the map  $\phi$  is continuous for the Zariski topology  $\mathcal{T}$  on  $\mathcal{X}$ .

**Proof.** Suppose  $Q \in \phi^{-1}(V^{\bar{R}}(\bar{I}))$ . Then  $\phi(Q) \in V^{\bar{R}}(\bar{I})$  and so  $\sqrt{(Q : M)} \supseteq I$ . Hence  $\sqrt{(Q : M)} \supseteq \sqrt{(IM : M)}$ . Thus  $Q \in \nu(IM)$ . The argument is reversible and so  $\phi$  is continuous.

**Theorem 2.2.** Let  $M$  be an  $R$ -module and the map  $\phi$  be surjective. Then  $\phi(\nu(N)) = V^{\overline{R}}(\overline{(N : M)})$  and  $\phi(\mathcal{X} - \nu(N)) = X^{\overline{R}} - V^{\overline{R}}(\overline{(N : M)})$  for every submodule  $N$  of  $M$ , i.e.,  $\phi$  is both closed and open.

**Proof.** As we have seen in Proposition 2.1,  $\phi$  is a continuous map such that  $\phi^{-1}(V^{\overline{R}}(\overline{I})) = \nu(IM)$ , for every ideal  $I \in V(Ann(M))$ . Hence for every submodule  $N$  of  $M$  we have  $\phi^{-1}(V^{\overline{R}}(\overline{(N : M)})) = \nu((N : M)M) = \nu(N)$ . Thus  $\phi(\nu(N)) = \phi \circ \phi^{-1}(V^{\overline{R}}(\overline{(N : M)})) = V^{\overline{R}}(\overline{(N : M)})$  as  $\phi$  is surjective. Similarly,  $\phi(\mathcal{X} - \nu(N)) = \phi(\phi^{-1}(X^{\overline{R}}) - \phi^{-1}(V^{\overline{R}}(\overline{(N : M)}))) = X^{\overline{R}} - V^{\overline{R}}(\overline{(N : M)})$ .

**Lemma 2.3.** Let  $M$  be an  $R$ -module. Then  $\phi(\mathcal{X}_r) \subseteq D_{\overline{r}}$  and the equality holds if  $\phi$  is surjective.

**Proof.** By Proposition 2.1,  $\phi^{-1}(D_{\overline{r}}) = \phi^{-1}(X^{\overline{R}} - V(\overline{Rr})) = \mathcal{X} - \nu(rM) = \mathcal{X}_r$ . Hence  $\phi(\mathcal{X}_r) \subseteq D_{\overline{r}}$ . The equality follows from Theorem 2.2.

**Lemma 2.4.** Let  $r, s \in R$ . Then the following hold.

- (1)  $\mathcal{X}_{rs} = \mathcal{X}_r \cap \mathcal{X}_s$ .
- (2)  $\mathcal{X}_{r^n} = \mathcal{X}_r$  for all  $n \in N$ .
- (3) If  $r$  is nilpotent, then  $\mathcal{X}_r = \emptyset$ .

**Proof.** (1) by Proposition 2.1,  $\mathcal{X}_{rs} = \phi^{-1}(D_{\overline{rs}})$ . Hence  $\mathcal{X}_{rs} = \phi^{-1}(D_{\overline{r}}) \cap \phi^{-1}(D_{\overline{s}}) = \mathcal{X}_r \cap \mathcal{X}_s$ .

(2) Follows from (1). (3) Since  $r$  is nilpotent,  $r^n = 0$  for some  $n \in N$ . Hence by (2),  $\mathcal{X}_r = \mathcal{X}_{r^n} = \mathcal{X}_0 = \emptyset$ .

For the remainder of this article, we assume that every  $R$ -module  $M$  is faithful and the map  $\phi$  is surjective except Lemmas 2.8 and 2.9 and Theorem 2.4.

**Lemma 2.5.** Let  $r, s \in R$  and  $M$  be an  $R$ -module. If  $\mathcal{X}_s \subseteq \mathcal{X}_r$ , then  $s \in \sqrt{\langle r \rangle}$ .

**Proof.** Suppose  $\mathcal{X}_s \subseteq \mathcal{X}_r$ . Hence  $\phi(\mathcal{X}_s) \subseteq \phi(\mathcal{X}_r)$ . Since  $\phi$  is surjective,  $D_{\overline{s}} \subseteq D_{\overline{r}}$ , by Lemma 2.3. Therefore  $D_s \subseteq D_r$  since  $M$  is faithful. Thus  $s \in \sqrt{\langle r \rangle}$ .

**Lemma 2.6.** Let  $r, s \in R$  and  $M$  be an  $R$ -module. If  $\mathcal{X}_s \subseteq \mathcal{X}_r$ , then there exists an  $R$ -module homomorphism  $\varphi_{r,s} : M_r \rightarrow M_s$ .

**Proof.** Since  $\mathcal{X}_s \subseteq \mathcal{X}_r$ , by Lemma 2.5  $s^n = tr$  for some positive integer  $n$  and  $t \in R$ . Hence  $\varphi_{r,s} : M_r \rightarrow M_s$  given by  $\frac{m}{r^k} \mapsto \frac{t^k m}{s^{nk}}$ . It is easy to see that  $\varphi_{r,s}$  is well-defined and an  $R$ -module homomorphism.

**Lemma 2.7.** Let  $r, s, t \in R$  and  $M$  be an  $R$ -module. If  $\mathcal{X}_t \subseteq \mathcal{X}_s \subseteq \mathcal{X}_r$ , then  $\varphi_{s,t} \circ \varphi_{r,s} = \varphi_{r,t}$ .

**Proof.** By Lemma 2.5,  $t^{n_1} = t_1 s$  and  $s^{n_2} = t_2 r$  for some positive integers  $n_1, n_2$  and  $t_1, t_2 \in R$ . Hence  $t^{n_1 n_2} = t_1^{n_2} s^{n_2} = t_1^{n_2} t_2 r$ . Now we have  $\varphi_{r,t}(\frac{m}{r^n}) = \frac{t_1^{n_2} t_2^n m}{t^{n n_1 n_2}} = \varphi_{s,t} \circ \varphi_{r,s}(\frac{m}{r^n})$ .

Let  $M$  be an  $R$ -module. Then for  $Q \in \mathcal{X}$  we set  $F_Q = \{\mathcal{X}_r \mid Q \in \mathcal{X}_r\}$ . Define an order in  $F_Q$  as follows. For  $\mathcal{X}_r$  and  $\mathcal{X}_s$ , write  $\mathcal{X}_r \leq \mathcal{X}_s$  when  $\mathcal{X}_s \subseteq \mathcal{X}_r$ . For arbitrary elements  $\mathcal{X}_{t_1}$  and  $\mathcal{X}_{t_2}$  in  $F_Q$ , we have  $\mathcal{X}_{t_1} \cap \mathcal{X}_{t_2} = \mathcal{X}_{t_1 t_2}$ . Then we get  $\mathcal{X}_{t_1} \leq \mathcal{X}_{t_1 t_2}$  and  $\mathcal{X}_{t_2} \leq \mathcal{X}_{t_1 t_2}$  even though there may not be an order between  $\mathcal{X}_{t_1}$  and  $\mathcal{X}_{t_2}$ . For an element  $\mathcal{X}_r$  in  $F_Q$  there corresponds an  $R$ -module  $M_r$  by Corollary 2.1 and that for  $\mathcal{X}_r \leq \mathcal{X}_s$ , there is an induced  $R$ -module homomorphism  $\varphi_{r,s} : M_r \rightarrow M_s$ . Then we defined the inductive limit  $\text{Lim}_{\mathcal{X}_r \in F_Q} M_r$  of  $\{M_r, \varphi_{r,s}\}$  as follows:

Let  $\Sigma = \{(\frac{m}{r^n}, \mathcal{X}_r) \mid \mathcal{X}_r \in F_Q\}$  and define  $(\frac{m}{r^n}, \mathcal{X}_r) \sim (\frac{m'}{s^{n_2}}, \mathcal{X}_s)$  if and only if there exists  $\mathcal{X}_t \in F_N$  such that  $\mathcal{X}_r \leq \mathcal{X}_t$  and  $\mathcal{X}_s \leq \mathcal{X}_t$  satisfying  $\varphi_{r,t}(\frac{m}{r^{n_1}}) = \varphi_{s,t}(\frac{m'}{s^{n_2}})$ . Then  $\text{Lim}_{\mathcal{X}_r \in F_Q} M_r$  is the quotient set  $\Sigma / \sim$ .

For  $[(a, \mathcal{X}_r)], [(b, \mathcal{X}_s)] \in \text{Lim}_{\mathcal{X}_r \in F_Q} M_r$ , let  $t = rs$ . Then  $\mathcal{X}_r \leq \mathcal{X}_t$  and  $\mathcal{X}_s \leq \mathcal{X}_t$ . Hence  $[(a, \mathcal{X}_r)] = [(\varphi_{r,t}(a), \mathcal{X}_t)]$  and  $[(b, \mathcal{X}_s)] = [(\varphi_{s,t}(b), \mathcal{X}_t)]$ . Then we define

$$[(a, \mathcal{X}_r)] + [(b, \mathcal{X}_s)] = [(\varphi_{r,rs}(a) + \varphi_{s,rs}(b), \mathcal{X}_{rs})] \text{ and} \\ [(t, D_r)][(m, \mathcal{X}_s)] = [(\rho_{D_{rs}, D_r}(t)\varphi_{s,rs}(m), \mathcal{X}_{rs})].$$

where  $\phi^{-1}(D_r) = \mathcal{X}_r$  and  $\rho_{D_{rs}, D_r} : R_r \rightarrow R_{rs}$ .

**Proposition 2.2.** Let  $M$  be an  $R$ -module. If  $\rho_{D_{rs}, D_r}(a) = \rho_{D_{trs}, D_r}(a)$  for all  $r, s, t, a \in R$ , then  $\text{Lim}_{\mathcal{X}_r \in F_Q} M_r$  is a  $\text{lim}_{D_r \in \nu_p} R_r$ -module, where  $p = \sqrt{(Q : M)}$  and  $\nu_p = \{D_r \mid p \in D_r\}$ .

**Proof.** It sufficient we consider the action  $\text{lim}_{D_r \in \nu_p} R_r$ -module  $\text{Lim}_{\mathcal{X}_r \in F_Q} M_r$  by  $[(t, D_r)][(m, \mathcal{X}_s)] = [(\rho_{D_{rs}, D_r}(t)\varphi_{s,rs}(m), \mathcal{X}_{rs})]$ .

**Theorem 2.3.** Let  $M$  be an  $R$ -module,  $Q \in \mathcal{X}$  and  $p = \sqrt{(Q : M)}$ . Then there exists an  $R_p$ -module isomorphism

$$\text{Lim}_{\mathcal{X}_r \in F_Q} M_r \cong M_p.$$

**Proof.** Assume  $\mu : M_p \rightarrow \text{Lim}_{\mathcal{X}_r \in F_Q} M_r$  by  $\mu(\frac{m}{r}) = [(\frac{m}{r}, \mathcal{X}_r)]$ . It is easy to see that  $\mu$  is well-defined. We show that  $\mu$  is an  $R_p$ -module homomorphism.

$$\mu(\frac{m_1}{r_1} + \frac{m_2}{r_2}) = \mu(\frac{m_1 r_2 + m_2 r_1}{r_1 r_2}) = [(\frac{m_1 r_2 + m_2 r_1}{r_1 r_2}, \mathcal{X}_{r_1 r_2})]$$

and

$$\mu(\frac{m_1}{r_1}) + \mu(\frac{m_2}{r_2}) = [(\frac{m_1}{r_1}, \mathcal{X}_{r_1})] + [(\frac{m_2}{r_2}, \mathcal{X}_{r_2})] = [(\varphi_{r_1, r_1 r_2}(\frac{m_1}{r_1}) + \varphi_{r_2, r_1 r_2}(\frac{m_2}{r_2}), \mathcal{X}_{r_1 r_2})] \\ = [(\frac{m_1 r_2}{r_1 r_2} + \frac{m_2 r_1}{r_1 r_2}, \mathcal{X}_{r_1 r_2})] = [(\frac{m_1 r_2 + m_2 r_1}{r_1 r_2}, \mathcal{X}_{r_1 r_2})].$$

Suppose  $\frac{m}{r} \in M_p$  and  $\frac{s}{t} \in R_p$ . Then  $\mu(\frac{s}{t} \cdot \frac{m}{r}) = \mu(\frac{sm}{tr}) = [(\frac{sm}{tr}, \mathcal{X}_{tr})]$ . Since  $\text{lim}_{D_r \in \nu_p} R_r \simeq R_p$ . Hence  $\frac{s}{t} \cdot \mu(\frac{m}{r}) = [(\frac{s}{t}, D_t)][(\frac{m}{r}, \mathcal{X}_r)] = [(\frac{sr}{tr}, D_{tr})][(\frac{mt}{rt}, \mathcal{X}_{rt})] = [(\frac{sr}{tr} \cdot \frac{mt}{rt}, \mathcal{X}_{tr})] = [(\frac{sm}{tr}, \mathcal{X}_{tr})]$ . Thus  $\mu$  is an  $R_p$ -module homomorphism.

We can define  $\theta : \text{Lim}_{\mathcal{X}_r \in F_Q} M_r \rightarrow M_p$  by  $\theta([(\frac{m}{r^n}, \mathcal{X}_r)]) = \frac{m}{r^n}$ . It is clear that  $\theta$  is well-defined. We show that  $\theta$  is an  $R_p$ -module homomorphism. We can easily show that  $\theta((\frac{m_1}{r_1}, \mathcal{X}_{r_1}) + (\frac{m_2}{r_2}, \mathcal{X}_{r_2})) = \theta((\frac{m_1}{r_1}, \mathcal{X}_{r_1})) + \theta((\frac{m_2}{r_2}, \mathcal{X}_{r_2}))$  for every  $r_1, r_2 \in R \setminus p$  and  $m_1, m_2 \in M$ . Assume  $[(\frac{m_1}{r_1}, \mathcal{X}_{r_1})] \in \text{Lim}_{\mathcal{X}_r \in F_Q} M_r$  and  $\frac{s}{t} \in R_p$ . Therefore  $\theta(\frac{s}{t} \cdot [(\frac{m_1}{r_1}, \mathcal{X}_{r_1})]) = \theta([(\frac{s}{t}, D_t)][(\frac{m_1}{r_1 t}, \mathcal{X}_{r_1 t})]) = \theta([(\frac{m_1 s}{r_1 t}, \mathcal{X}_{r_1 t})]) = \frac{m_1 s}{r_1 t} = \frac{s}{t} \cdot \theta([(\frac{m_1}{r_1}, \mathcal{X}_{r_1})])$ . Suppose  $\frac{m}{r} \in M_p$ . Hence  $(\theta \circ \mu)(\frac{m}{r}) = \frac{m}{r}$ . On the other hand, let  $[(\frac{s}{r^n}, \mathcal{X}_r)] \in \text{Lim}_{\mathcal{X}_r \in F_Q} M_r$ . Then  $(\mu \circ \theta)([(\frac{s}{r^n}, \mathcal{X}_r)]) = \mu(\frac{s}{r^n}) = [(\frac{s}{r^n}, \mathcal{X}_{r^n})] = [(\frac{s}{r^n}, \mathcal{X}_r)]$  since  $\mathcal{X}_{r^n} = \mathcal{X}_r$ . Consequently,  $\mu$  and  $\theta$  are isomorphisms.

The saturation of a submodule  $N$  of an  $R$ -module  $M$  with respect to a prime ideal  $p$  of  $R$  is the contraction of  $N_p$  in  $M$  and designated by  $S_p(N)$ . It is known that  $S_p(N) = \{m \in M \mid rm \in N \text{ for some } r \in R \setminus p\}$ . [6]

**Lemma 2.8.** Let  $Q \in \mathcal{X}$ . Then  $S_p(Q) \subseteq \text{rad } Q$  for every  $p \in V((Q : M))$ . In particular, if  $S_p(Q)$  is a prime submodule of  $M$  for some  $p \in V((Q : M))$ , then  $S_p(Q) = \text{rad } Q$ .

**Proof.** Straightforward.

**Lemma 2.9.** Let  $Q \in \mathcal{X}$ . Consider the following statements.

- (1)  $\text{rad}Q$  is a  $p$ -prime submodule of  $M$ .
- (2)  $\text{rad}Q$  is a  $p$ -primary-like submodule of  $M$ .

Then (1)  $\Leftrightarrow$  (2).

Furthermore, if  $(Q : M) = p$  is a radical ideal of  $R$  and

- (3)  $Q$  is a  $p$ -primary-like submodule of  $M$ ,

then (1) – (3) are equivalent.

**Proof.** (3)  $\Rightarrow$  (1) Since  $S_p(Q) \subseteq \text{rad}Q$ , then  $S_p(Q) \neq M$ . Thus by [6] and Lemma 2.8,  $\text{rad}Q$  is prime. The verification of the other implications is straightforward.

**Theorem 2.4.** Let  $M$  be an  $R$ -module and  $Q \in \mathcal{X}$ . Then for each the following cases  $\text{rad}Q$  is prime.

- (1)  $R$  is a zero-dimensional ring.
- (2) For each  $Q \in \mathcal{X}$  and prime ideal  $p = \sqrt{(Q : M)}$ ,  $(S_p(Q) : M)$  is a radical ideal.
- (3) For each  $Q \in \mathcal{X}$  and prime ideal  $p = \sqrt{(Q : M)}$ ,  $S_p(Q) \neq M$ .
- (4)  $M$  is a multiplication module.
- (5)  $R$  is a Noetherian domain and  $Q \in \mathcal{X}$  is contained in only finitely many prime submodules of  $M$ .

**Proof.** (1) Suppose  $Q \in \mathcal{X}$ . Since  $R$  is zero-dimensional,  $\sqrt{(Q : M)} = (P : M)$  for all prime submodules  $P$  containing  $Q$ . So  $p = \sqrt{(Q : M)} = (\text{rad}Q : M)$  is a prime ideal of  $R$ . Now, if  $rm \in \text{rad}Q$  and  $m \notin \text{rad}Q$ , there is a prime submodule  $P$  containing  $Q$  such that  $rm \in P$  and  $m \notin P$  and so  $r \in (P : M) = \sqrt{(Q : M)} = (\text{rad}Q : M)$ . Thus  $\text{rad}Q$  is prime.

(2)  $p = \sqrt{(Q : M)} \subseteq \sqrt{(S_p(Q) : M)} \subseteq (\text{rad}Q : M) = \sqrt{(Q : M)} = p$ . It follows that  $\sqrt{(S_p(Q) : M)} = p$ . Now since  $(S_p(Q) : M)$  is a radical ideal, we have  $(S_p(Q) : M) = p$ . It follows from [6] and Lemma 2.8,  $\text{rad}Q$  is a prime submodule.

(3) Suppose  $S_p(Q) \neq M$ . By [6],  $S_p(Q)$  is a prime submodule of  $M$ . It follows from Lemma 2.8,  $\text{rad}Q$  is a prime submodule of  $M$ .

(4) Since for every  $Q \in \mathcal{X}$ ,  $\text{rad}Q$  is proper and  $(\text{rad}Q : M)$  is prime,  $\text{rad}Q$  is prime by [1].

(5) By Lemma 2.9 we may assume that  $(Q : M) \neq 0$ . If  $P$  is a prime submodule containing  $Q$ , then  $0 \subset \sqrt{(Q : M)} \subseteq (P : M)$  is a chain of prime ideals of  $R$ . If the later containment is proper, by [4] there are infinitely many prime ideals  $p$  with  $(Q : M) \subset p \subset (P : M)$  and so we have infinitely prime submodules  $P$  containing  $Q$ , a contradiction. Hence we have  $\sqrt{(Q : M)} = (P : M)$ , for all prime submodules  $P$  containing  $N$ . Now, if  $rm \in \text{rad}Q$  and  $m \notin \text{rad}Q$ , there is a prime submodule  $P$  containing  $Q$  such that  $rm \in P$  and  $m \notin P$  and so that  $r \in (P : M) = \sqrt{(Q : M)} = (\text{rad}Q : M)$ .

**Corollary 2.2.** Let  $M$  be an  $R$ -module,  $Q \in \mathcal{X}$  and  $p = \sqrt{(Q : M)}$ . Then there exists an  $R_p$ -module isomorphism

$$\text{Lim}_{\mathcal{X}_r \in F_Q} M_r \cong \text{lim}_{\mathcal{X}_r \in \nu_{\text{rad}Q}} M_r.$$



**Proof.** Combine Theorem 2.3 and [10].

**Lemma 2.10.** If  $\mathcal{X}_r = \cup_{\lambda \in \Lambda} \mathcal{X}_{r_\lambda}$  and for  $m \in M_r$ ,  $\varphi_{r, r_\lambda}(m) = 0$ ,  $\lambda \in \Lambda$ , then  $m = 0$ .

**Proof.** Suppose  $m = \frac{m'}{r^n} \in M_r$ . If  $m = 0$ , then there exists a positive integer  $n'$  such that  $r^{n'} m' = 0$ . Put  $I = \{s \in R \mid sm' = 0\}$ . Then  $I$  is an ideal of  $R$ . Therefore,  $m = 0$  if and only if  $r \in \sqrt{I}$ . Therefore  $r \in p$  for every prime ideal  $p$  of  $R$  containing  $I$ . Since  $\phi$  is surjective,  $p = \sqrt{(Q : M)}$  for some  $Q \in \mathcal{X}$ . Suppose  $m \neq 0$  in  $M_r$ . Then there exists a prime ideal  $p$  of  $R$  containing  $I$  such that  $r \notin p$ , where  $p = \sqrt{(Q : M)}$ . Hence  $Q \in \mathcal{X}_r$ . Since  $\mathcal{X}_r = \cup_{\lambda \in \Lambda} \mathcal{X}_{r_\lambda}$ , there exists  $r_\lambda$  such that  $Q \in \mathcal{X}_{r_\lambda}$  and so  $r_\lambda \notin p$ . Since  $\varphi_{r, r_\lambda}(m) = 0$  the image of  $m$  in  $M_{r_\lambda}$  is 0. Therefore  $\frac{m' a^{n'}}{r_\lambda^{n n'}} = \frac{0}{r_\lambda}$ , where  $a \in R$  and  $r_\lambda \notin p$ . Then  $r_\lambda^n a^{n'} m' = 0$ . Hence there exists  $b = r_\lambda^n a^{n'} \in R \setminus p$  such that  $bm' = 0$ . Thus  $b \in I \subseteq p$ , a contradiction.

**Lemma 2.11.** Let  $\mathcal{X} = \cup_{\lambda \in \Lambda} \mathcal{X}_{r_\lambda}$ . Then the ideal generated by  $\{r_\lambda\}$  equals to  $R$ .

**Proof.** Suppose  $\mathcal{X} = \cup_{\lambda \in \Lambda} \mathcal{X}_{r_\lambda}$ . Since  $\phi$  is surjective,  $\text{Spec}(R) = \phi(\mathcal{X}) = \phi(\cup_{\lambda \in \Lambda} \mathcal{X}_{r_\lambda}) = \cup_{\lambda \in \Lambda} (\phi(\mathcal{X}_{r_\lambda})) = \cup_{\lambda \in \Lambda} D_{r_\lambda}$  by Lemma 2.3. Hence  $\text{Spec}(R) = \cup_{\lambda \in \Lambda} (\text{Spec}(R) - V(Rr_\lambda)) = \text{Spec}(R) - \cap_{\lambda \in \Lambda} V(Rr_\lambda) = \text{Spec}(R) - V(\sum_{\lambda \in \Lambda} Rr_\lambda)$ . Therefore  $V(\sum_{\lambda \in \Lambda} Rr_\lambda) = \emptyset$ . Thus  $\sum_{\lambda \in \Lambda} Rr_\lambda = R$ .

**Proposition 2.3.** Given that  $\mathcal{X}_r = \cup_{\lambda \in \Lambda} \mathcal{X}_{r_\lambda}$  and that, for  $m_\lambda \in M_{r_\lambda}$ ,  $\lambda \in \Lambda$ , and for arbitrary  $\lambda, \lambda' \in \Lambda$ ,  $\varphi_{r_\lambda, r_\lambda r_{\lambda'}}(m_\lambda) = \varphi_{r_{\lambda'}, r_\lambda r_{\lambda'}}(m_{\lambda'})$ , then there exists  $m \in M_r$  satisfying  $m_\lambda = \varphi_{1, r_\lambda}(m)$ ,  $\lambda \in \Lambda$ .

**Proof.** Let  $\sigma_r : R \rightarrow R_r$  define by  $s \rightarrow \frac{sr}{s}$  for all  $s \in R$ . Let us write  $\sigma_r(s)$  simply as  $\bar{s}$ . When  $\mathcal{X}_t \subseteq \mathcal{X}_r$ , then  $t \in \sqrt{\langle r \rangle}$  by Lemma 2.5. Therefore, there exists an  $R$ -module isomorphism from  $M_t$  to  $(M_r)_{\bar{t}}$ . Let  $\bar{M} = M_r$  and  $\bar{\mathcal{X}} = \text{Spec}_L(\bar{M})$  and let  $\bar{r}_\lambda$  be the element in  $\bar{R}$  determined by  $r_\lambda$ . Then  $\mathcal{X}_r = \bar{\mathcal{X}}$  and  $\mathcal{X}_{r_\lambda} = \bar{\mathcal{X}}_{\bar{r}_\lambda}$ . Therefore, we can assume  $r = 1$  without loss of generality. If we let  $r = 1$ , our assumption  $\mathcal{X} = \cup_{\lambda \in \Lambda} \mathcal{X}_{r_\lambda}$ . From Lemma 2.11, one can choose finitely many  $r_1, r_2, \dots, r_n$  among  $\{r_\lambda\}_{\lambda \in \Lambda}$  such that  $\mathcal{X} = \bigcup_{i=1}^n \mathcal{X}_{r_i}$ . From the hypothesis, we have  $\varphi_{r_j, r_j r_i}(m_j) = \frac{r_i^n m'_j}{(r_i r_j)^n} = \varphi_{r_i, r_j r_i}(m_i) = \frac{r_j^n m'_i}{(r_i r_j)^n}$ . Therefore, one can suppose a non-negative integer  $t_{ji}$  such that  $(r_j r_i)^{t_{ji}} (r_i^n m'_j - r_j^n m'_i) = 0$ ,  $1 \leq j < i \leq n$ . Then let  $k > n + t_{ji}$  for all  $1 \leq j < i \leq n$ . Since for every  $1 \leq h \leq n$  we can write  $m_h = \frac{m'_h}{r_h^n} = \frac{r_h^{k-n} m'_h}{r_h^k}$ . We set  $m''_h = r_h^{k-n} m'_h$ . Hence  $m_h = \frac{m''_h}{r_h^k}$ . Also we have  $m''_j r_i^k - m''_i r_j^k = (r_j^{k-n} m'_j) r_i^k - (r_i^{k-n} m'_i) r_j^k = (r_j r_i)^k r_j^{-n} m'_j - (r_j r_i)^k r_i^{-n} m'_i = (r_j r_i)^{n+t_{ji}} (r_j r_i)^s r_j^{-n} m'_j - (r_j r_i)^{n+t_{ji}} (r_j r_i)^s r_i^{-n} m'_i = (r_j r_i)^{t_{ji}} [(r_j r_i)^s r_i^n m'_j - (r_j r_i)^s r_j^n m'_i] = (r_j r_i)^s [(r_j r_i)^{t_{ji}} (r_i^n m'_j - r_j^n m'_i)] = 0$ ,  $1 \leq j < i \leq n$ . Since  $\mathcal{X}_{r_i} = \mathcal{X}_{r_i^k}$ , then  $\mathcal{X} = \cup_{i=1}^n \mathcal{X}_{r_i^k}$ . Hence by Lemma 2.11, there exists  $a_i \in R$  such that  $\sum_{i=1}^n a_i r_i^k = 1$ . Assume  $m = \sum_{i=1}^n a_i m''_i \in M$ . Hence  $r_j^k m = \sum_{i=1}^n a_i r_j^k m''_i = \sum_{i=1}^n a_i r_i^k m''_j = m''_j$ . Thus  $\varphi_{1, r_j}(m) = \frac{m''_j}{r_j^k} = m_j$ . On the other hand, for arbitrary  $\lambda \in \Lambda$ , we put  $b_\lambda = m_\lambda \varphi_{1, r_\lambda}(m)$ . Then for any  $j$  we have  $\varphi_{r_\lambda, r_j r_\lambda}(b_\lambda) = \varphi_{r_\lambda, r_j r_\lambda}(m_\lambda) - \varphi_{1, r_j r_\lambda}(m) = \varphi_{r_j, r_j r_\lambda}(m_j) - \varphi_{r_\lambda, r_j r_\lambda}(m) = \varphi_{r_j, r_j r_\lambda}(\varphi_{r_\lambda, r_j r_\lambda}(m)) - \varphi_{r_\lambda, r_j r_\lambda}(m) = \varphi_{r_\lambda, r_j r_\lambda}(m) - \varphi_{r_\lambda, r_j r_\lambda}(m) = 0$ . Hence  $b_\lambda = 0$ , by Lemma 2.10. Thus  $m_\lambda = \varphi_{1, r_\lambda}(m)$  for  $\lambda \in \Lambda$ .

Let  $M$  be an  $R$ -module. We will define the sheaf  $\mathcal{O}_{\mathcal{X}}$  of  $R$ -modules over  $(\mathcal{X}, \mathcal{T})$ . Let  $\mathcal{U}$  be an arbitrary open set of  $\mathcal{X}$ . Define  $\mathcal{O}_{\mathcal{X}}(\mathcal{U})$  abruptly as a subset of  $\prod_{Q \in \mathcal{U}, \sqrt{Q}: M=p} M_p$  where  $p$  is a

prime ideal of  $R$ :

$$\mathcal{O}_{\mathcal{X}}(\mathcal{U}) = \left\{ \begin{array}{l} \{m_p\} \in \prod_{Q \in \mathcal{U}, \sqrt{Q:M}=p} M_p : \text{If an open covering } \{\mathcal{X}_{r_\lambda}\}_{\lambda \in \Lambda} \text{ of } \mathcal{U} \text{ and} \\ m_\lambda \in M_{r_\lambda} \text{ are chosen properly, then for } Q \in \mathcal{X}_\lambda, \text{ the germ } m_\lambda \\ \text{at } Q \text{ coincides with } m_p \text{ where } \sqrt{(Q:M)} = p \end{array} \right\}.$$

**Lemma 2.12.** Let  $m_\lambda \in M_{r_\lambda}$  and  $m_{\lambda'} \in M_{r_{\lambda'}}$ . Then  $\varphi_{r_\lambda, r_\lambda r_{\lambda'}}(m_\lambda) = \varphi_{r_{\lambda'}, r_\lambda r_{\lambda'}}(m_{\lambda'})$ .

**Proof.** The germ of  $m_\lambda$  at  $Q \in \mathcal{X}_{r_\lambda r_{\lambda'}}$  equaling the germ of  $m_{\lambda'}$  at  $Q$  means that the inductive limit; namely, for a certain  $\mathcal{X}_{h_Q} \subseteq \mathcal{X}_{r_\lambda r_{\lambda'}}$  containing  $Q$ ,  $(*) \varphi_{r_\lambda, r_{\lambda'}}(m_\lambda) = \varphi_{r_{\lambda'}, r_{\lambda'}}(m_{\lambda'})$ . Then for each  $Q \in \mathcal{X}_{r_\lambda r_{\lambda'}}$ , choose an open set  $\mathcal{X}_{h_Q}$  satisfying  $(*)$ . We get  $\mathcal{X}_{r_\lambda r_{\lambda'}} = \bigcup_{Q \in \mathcal{X}_{r_\lambda r_{\lambda'}}} \mathcal{X}_{h_Q}$ . Then, from  $(*)$  for all  $Q \in \mathcal{X}_{r_\lambda r_{\lambda'}}$  we have  $\varphi_{r_{\lambda'}, h_Q}(\varphi_{r_\lambda, h_Q}(m_\lambda)) = \varphi_{r_{\lambda'}, h_Q}(\varphi_{r_{\lambda'}, h_Q}(m_{\lambda'}))$ . Therefore  $\varphi_{r_{\lambda'}, h_Q}(\varphi_{r_\lambda, h_Q}(m_\lambda) - \varphi_{r_{\lambda'}, h_Q}(m_{\lambda'})) = 0$ . Thus  $\varphi_{r_\lambda, r_{\lambda'}}(m_\lambda) = \varphi_{r_{\lambda'}, r_{\lambda'}}(m_{\lambda'})$  by Lemma 2.10.

Let  $X$  be a topological space, and for an open subset  $U$  of  $X$ , let  $F(U)$  be an  $R$ -module. Then  $F$  is said to be a sheaf if the following conditions are satisfied.<sup>[3]</sup>

- (1) For an open set  $V \subseteq U$  of  $X$ , there exists an  $R$ -module homomorphism  $\rho_{V,U}^* : F(U) \rightarrow F(V)$  such that:

- (i)  $\rho_{U,U}^* = id_{F(U)}$ ,
- (ii) for open sets  $W \subseteq V \subseteq U$  of  $X$ ,  $\rho_{W,U}^* = \rho_{W,V}^* \circ \rho_{V,U}^*$ .

- (2) Let  $U$  be an open set in  $X$  such that  $U$  is a union of open sets, i.e.,  $U = \bigcup_{j \in J} U_j$ .

- (i) If  $m \in F(U)$  satisfies  $\rho_{U_j,U}^*(m) = 0$ ,  $j \in J$ , then  $m = 0$ .
- (ii) If  $m_j \in F(U_j)$ ,  $j \in J$ , satisfies  $\rho_{U_i \cap U_j, U_j}^*(m_j) = \rho_{U_i \cap U_j, U_i}^*(m_i)$ ,  $i, j \in J$ , then there exists  $m \in F(U)$   $m_j = \rho_{U_j, U}^*(m)$ .

Let  $M$  be an  $R$ -module. It is easily seen that  $\mathcal{O}_{\mathcal{X}}(\mathcal{U})$  with addition and multiplication as  $\{m_p\} + \{m'_p\} = \{m_p + m'_p\}$  and  $r\{m_p\} = \{rm_p\}$ , where  $r \in R$ ,  $m, m' \in M$  and  $p$  is prime ideal of  $R$ , is an  $R$ -module. Furthermore, for open sets  $\mathcal{V} \subseteq \mathcal{U}$  we define  $\varphi_{\mathcal{U}, \mathcal{V}} : \mathcal{O}_{\mathcal{X}}(\mathcal{U}) \rightarrow \mathcal{O}_{\mathcal{X}}(\mathcal{V})$  given by  $\{m_p\}_{Q \in \mathcal{U}} \mapsto \{m'_p\}_{Q' \in \mathcal{V}}$ , where  $p = \sqrt{(Q:M)}$  and  $p' = \sqrt{(Q':M)}$ .

**Theorem 2.5.** Let  $M$  be an  $R$ -module. Then  $\mathcal{O}_{\mathcal{X}}$  is a sheaf of  $R$ -modules over  $\mathcal{X}$  with the Zariski topology  $\mathcal{T}$  and  $\mathcal{O}_{\mathcal{X}}(\mathcal{X}) = M$ .

**Proof.** It is easy to see that the condition (1) of the above definition hold. Now we check the condition (2). Suppose that  $\mathcal{U} = \bigcup_{i \in I} \mathcal{U}_i$  and  $m \in \mathcal{O}_{\mathcal{X}}(\mathcal{U})$  such that  $\varphi_{\mathcal{U}, \mathcal{U}_i}(m) = 0$ ,  $i \in I$ . Since  $m \in \mathcal{O}_{\mathcal{X}}(\mathcal{U})$ , then  $m = \{m_p\} \in \prod_{Q \in \mathcal{U}, \sqrt{Q:M}=p} M_p$ . Therefore  $0 = \varphi_{\mathcal{U}, \mathcal{U}_i}(m) = \{m_p\}_{Q \in \mathcal{U}_i}$ . Hence for every  $Q \in \mathcal{U}_i$  we have  $m_p = 0$ . Since  $\mathcal{U} = \bigcup_{i \in I} \mathcal{U}_i$ , then for every  $Q \in \mathcal{U}$  we have  $m_p = 0$ . Thus  $m = 0$ . Assume  $m_i \in \mathcal{O}_{\mathcal{X}}(\mathcal{U}_i)$  and  $\varphi_{\mathcal{U}_i, \mathcal{U}_i \cap \mathcal{U}_j}(m_i) = \varphi_{\mathcal{U}_j, \mathcal{U}_i \cap \mathcal{U}_j}(m_j)$ ,  $i, j \in I$ . Put  $m_i = \{m_p^{(i)}\} \in \prod_{Q \in \mathcal{U}_i, \sqrt{Q:M}=p} M_p$ . Therefore  $\{m_p^{(i)}\}_{Q \in \mathcal{U}_i \cap \mathcal{U}_j} = \varphi_{\mathcal{U}_i, \mathcal{U}_i \cap \mathcal{U}_j}(m_i) = \varphi_{\mathcal{U}_j, \mathcal{U}_i \cap \mathcal{U}_j}(m_j) = \{m_p^{(j)}\}_{Q \in \mathcal{U}_i \cap \mathcal{U}_j}$ . Hence for each  $Q \in \mathcal{U}_i \cap \mathcal{U}_j$  we have  $m_p^{(i)} = m_p^{(j)}$ . Now, let  $m = \{m_p\} \in \prod_{Q \in \mathcal{U}, \sqrt{Q:M}=p} M_p$ . For every  $Q \in \mathcal{U}_i$  we put  $m_p = m_p^{(i)}$ . Hence  $m \in \mathcal{O}_{\mathcal{X}}(\mathcal{U})$  and

$\varphi_{\mathcal{U}, \mathcal{U}_i}(m) = \{m_p\}_{Q \in \mathcal{U}_i} = m_i$ . Thus the condition (2) holds. It is clear that  $\mathcal{O}_{\mathcal{X}}(\mathcal{X}_r) = \mathcal{O}_{\mathcal{X}}(\mathcal{X}) = M$  for  $r = 1$ .

## References

- [1] Z. A. El-Bast and P. F. Smith, Multiplication modules, *Comm. Algebra*, **16**(1988), No. 4, 755-779.
- [2] H. Fazaeli Moghimi and F. Rashedi, Primary-like submodules satisfying the primeful property, (to appear).
- [3] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York Inc., 1977.
- [4] I. Kaplansky, *Commutative rings*, University of Chicago Press, Chicago, 1974.
- [5] C. P. Lu, A module whose prime spectrum has the surjective natural map, *Houston J. Math.*, **33**(2007), No. 1, 125-143.
- [6] C. P. Lu, Saturations of submodules, *Comm. Algebra*, **31**(2003), No. 6, 2655-2673.
- [7] C. P. Lu, The Zariski topology on the prime spectrum of a module, *Houston J. Math.*, **25**(1999), No. 3, 417-425.
- [8] C. P. Lu, Spectra of modules, *Comm. Algebra*, **23**(1995), No. 10, 3741-3752.
- [9] M. E. Moor and S. J. Smith, Prime and radical submodules of modules over commutative rings, *Comm. Algebra*, **30**(2002), 5073-5064.
- [10] U. Tekir, On the sheaf of modules, *Comm. Algebra*, **33**(2005), 2557-2562.

## Soft neutrosophic semigroup and their generalization

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**Abstract** Soft set theory is a general mathematical tool for dealing with uncertain, fuzzy, not clearly defined objects. In this paper we introduced soft neutrosophic semigroup, soft neutrosophic bisemigroup, soft neutrosophic  $N$ -semigroup with the discussion of some of their characteristics. We also introduced a new type of soft neutrosophic semigroup, the so called soft strong neutrosophic semigroup which is of pure neutrosophic character. This notion also found in all the other corresponding notions of soft neutrosophic theory. We also given some of their properties of this newly born soft structure related to the strong part of neutrosophic theory.

**Keywords** Neutrosophic semigroup, neutrosophic bisemigroup, neutrosophic  $N$ -semigroup, soft set, soft semigroup, soft neutrosophic semigroup, soft neutrosophic bisemigroup, soft neutrosophic  $N$ -semigroup.

### §1. Introduction and preliminaries

Florentine Smarandache for the first time introduced the concept of neutrosophy in 1995, which is basically a new branch of philosophy which actually studies the origin, nature, and scope of neutralities. The neutrosophic logic came into being by neutrosophy. In neutrosophic logic each proposition is approximated to have the percentage of truth in a subset  $T$ , the percentage of indeterminacy in a subset  $I$ , and the percentage of falsity in a subset  $F$ . Neutrosophic logic is an extension of fuzzy logic. In fact the neutrosophic set is the generalization of classical set, fuzzy conventional set, intuitionistic fuzzy set, and interval valued fuzzy set. Neutrosophic logic is used to overcome the problems of impreciseness, indeterminate, and inconsistencies of date etc. The theory of neutrosophy is so applicable to every field of algebra. W. B. Vasantha Kandasamy and Florentin Smarandache introduced neutrosophic fields, neutrosophic rings, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups and neutrosophic  $N$ -groups, neutrosophic semigroups, neutrosophic bisemigroups, and neutrosophic

$N$ -semigroups, neutrosophic loops, neutrosophic biloops, and neutrosophic  $N$ -loops, and so on. Mumtaz ali et al. introduced neutrosophic  $LA$ -semigroups.

Molodtsov introduced the theory of soft set. This mathematical tool is free from parameterization inadequacy, syndrome of fuzzy set theory, rough set theory, probability theory and so on. This theory has been applied successfully in many fields such as smoothness of functions, game theory, operation research, Riemann integration, Perron integration, and probability. Recently soft set theory attained much attention of the researchers since its appearance and the work based on several operations of soft set introduced in [2, 9, 10]. Some properties and algebra may be found in [1]. Feng et al. introduced soft semirings in [5]. By means of level soft sets an adjustable approach to fuzzy soft set can be seen in [6]. Some other concepts together with fuzzy set and rough set were shown in [7, 8].

This paper is about to introduced soft neutrosophic semigroup, soft neutrosophic group, and soft neutrosophic  $N$ -semigroup and the related strong or pure part of neutrosophy with the notions of soft set theory. In the proceeding section, we define soft neutrosophic semigroup, soft neutrosophic strong semigroup, and some of their properties are discussed. In the next section, soft neutrosophic bisemigroup are presented with their strong neutrosophic part. Also in this section some of their characterization have been made. In the last section soft neutrosophic  $N$ -semigroup and their corresponding strong theory have been constructed with some of their properties.

## §2. Definition and properties

**Definition 2.1.** Let  $S$  be a semigroup, the semigroup generated by  $S$  and  $I$  i.e.  $S \cup I$  denoted by  $\langle S \cup I \rangle$  is defined to be a neutrosophic semigroup where  $I$  is indeterminacy element and termed as neutrosophic element.

It is interesting to note that all neutrosophic semigroups contain a proper subset which is a semigroup.

**Example 2.1.** Let  $Z = \{\text{the set of positive and negative integers with zero}\}$ ,  $Z$  is only a semigroup under multiplication. Let  $N(S) = \{\langle Z \cup I \rangle\}$  be the neutrosophic semigroup under multiplication. Clearly  $Z \subset N(S)$  is a semigroup.

**Definition 2.2.** Let  $N(S)$  be a neutrosophic semigroup. A proper subset  $P$  of  $N(S)$  is said to be a neutrosophic subsemigroup, if  $P$  is a neutrosophic semigroup under the operations of  $N(S)$ . A neutrosophic semigroup  $N(S)$  is said to have a subsemigroup if  $N(S)$  has a proper subset which is a semigroup under the operations of  $N(S)$ .

**Theorem 2.1.** Let  $N(S)$  be a neutrosophic semigroup. Suppose  $P_1$  and  $P_2$  be any two neutrosophic subsemigroups of  $N(S)$  then  $P_1 \cup P_2$  (i.e. the union) the union of two neutrosophic subsemigroups in general need not be a neutrosophic subsemigroup.

**Definition 2.3.** A neutrosophic semigroup  $N(S)$  which has an element  $e$  in  $N(S)$  such that  $e * s = s * e = s$  for all  $s \in N(S)$ , is called as a neutrosophic monoid.

**Definition 2.4.** Let  $N(S)$  be a neutrosophic monoid under the binary operation  $*$ . Suppose  $e$  is the identity in  $N(S)$ , that is  $s * e = e * s = s$  for all  $s \in N(S)$ . We call a proper subset  $P$  of  $N(S)$  to be a neutrosophic submonoid if

1.  $P$  is a neutrosophic semigroup under  $*$ .
2.  $e \in P$ , i.e.,  $P$  is a monoid under  $*$ .

**Definition 2.5.** Let  $N(S)$  be a neutrosophic semigroup under a binary operation  $*$ .  $P$  be a proper subset of  $N(S)$ .  $P$  is said to be a neutrosophic ideal of  $N(S)$  if the following conditions are satisfied.

1.  $P$  is a neutrosophic semigroup.
2. For all  $p \in P$  and for all  $s \in N(S)$  we have  $p * s$  and  $s * p$  are in  $P$ .

**Definition 2.6.** Let  $N(S)$  be a neutrosophic semigroup.  $P$  be a neutrosophic ideal of  $N(S)$ ,  $P$  is said to be a neutrosophic cyclic ideal or neutrosophic principal ideal if  $P$  can be generated by a single element.

**Definition 2.7.** Let  $(BN(S), *, o)$  be a nonempty set with two binary operations  $*$  and  $o$ .  $(BN(S), *, o)$  is said to be a neutrosophic bisemigroup if  $BN(S) = P_1 \cup P_2$  where atleast one of  $(P_1, *)$  or  $(P_2, o)$  is a neutrosophic semigroup and other is just a semigroup.  $P_1$  and  $P_2$  are proper subsets of  $BN(S)$ , i.e.  $P_1 \subsetneq P_2$ .

If both  $(P_1, *)$  and  $(P_2, o)$  in the above definition are neutrosophic semigroups then we call  $(BN(S), *, o)$  a strong neutrosophic bisemigroup. All strong neutrosophic bisemigroups are trivially neutrosophic bisemigroups.

**Example 2.2.** Let  $(BN(S), *, o) = \{0, 1, 2, 3, I, 2I, 3I, S(3), *, o\} = (P_1, *) \cup (P_2, o)$  where  $(P_1, *) = \{0, 1, 2, 3, I, 2I, 3I, *\}$  and  $(P_2, o) = (S(3), o)$ . Clearly  $(P_1, *)$  is a neutrosophic semigroup under multiplication modulo 4.  $(P_2, o)$  is just a semigroup. Thus  $(BN(S), *, o)$  is a neutrosophic bisemigroup.

**Definition 2.8.** Let  $(BN(S) = P_1 \cup P_2; o, *)$  be a neutrosophic bisemigroup. A proper subset  $(T, o, *)$  is said to be a neutrosophic subbisemigroup of  $BN(S)$  if

1.  $T = T_1 \cup T_2$  where  $T_1 = P_1 \cap T$  and  $T_2 = P_2 \cap T$ .
2. At least one of  $(T_1, o)$  or  $(T_2, *)$  is a neutrosophic semigroup.

**Definition 2.9.** Let  $(BN(S) = P_1 \cup P_2, o, *)$  be a neutrosophic strong bisemigroup. A proper subset  $T$  of  $BN(S)$  is called the strong neutrosophic subbisemigroup if  $T = T_1 \cup T_2$  with  $T_1 = P_1 \cap T$  and  $T_2 = P_2 \cap T$  and if both  $(T_1, *)$  and  $(T_2, o)$  are neutrosophic subsemigroups of  $(P_1, *)$  and  $(P_2, o)$  respectively. We call  $T = T_1 \cup T_2$  to be a neutrosophic strong subbisemigroup, if atleast one of  $(T_1, *)$  or  $(T_2, o)$  is a semigroup then  $T = T_1 \cup T_2$  is only a neutrosophic subsemigroup.

**Definition 2.10.** Let  $(BN(S) = P_1 \cup P_2, *, o)$  be any neutrosophic bisemigroup. Let  $J$  be a proper subset of  $B(NS)$  such that  $J_1 = J \cap P_1$  and  $J_2 = J \cap P_2$  are ideals of  $P_1$  and  $P_2$  respectively. Then  $J$  is called the neutrosophic bi-ideal of  $BN(S)$ .

**Definition 2.11.** Let  $(BN(S), *, o)$  be a strong neutrosophic bisemigroup where  $BN(S) = P_1 \cup P_2$  with  $(P_1, *)$  and  $(P_2, o)$  be any two neutrosophic semigroups. Let  $J$  be a proper subset of  $BN(S)$  where  $I = I_1 \cup I_2$  with  $I_1 = J \cap P_1$  and  $I_2 = J \cap P_2$  are neutrosophic ideals of the neutrosophic semigroups  $P_1$  and  $P_2$  respectively. Then  $I$  is called or defined as the strong neutrosophic bi-ideal of  $B(N(S))$ .

Union of any two neutrosophic bi-ideals in general is not a neutrosophic bi-ideal. This is true of neutrosophic strong bi-ideals.

**Definition 2.12.** Let  $\{S(N), *_1, \dots, *_N\}$  be a non empty set with  $N$ -binary operations

defined on it. We call  $S(N)$  a neutrosophic  $N$ -semigroup ( $N$  a positive integer) if the following conditions are satisfied.

1.  $S(N) = S_1 \cup \dots \cup S_N$  where each  $S_i$  is a proper subset of  $S(N)$  i.e.  $S_i \subsetneq S_j$  or  $S_j \subsetneq S_i$  if  $i \neq j$ .

2.  $(S_i, *_i)$  is either a neutrosophic semigroup or a semigroup for  $i = 1, 2, \dots, N$ .

If all the  $N$ -semigroups  $(S_i, *_i)$  are neutrosophic semigroups (i.e. for  $i = 1, 2, \dots, N$ ) then we call  $S(N)$  to be a neutrosophic strong  $N$ -semigroup.

**Example 2.3.** Let  $S(N) = \{S_1 \cup S_2 \cup S_3 \cup S_4, *_1, *_2, *_3, *_4\}$  be a neutrosophic 4-semigroup where

$S_1 = \{Z_{12}, \text{semigroup under multiplication modulo } 12\}$ .

$S_2 = \{0, 1, 2, 3, I, 2I, 3I, \text{semigroup under multiplication modulo } 4\}$ , a neutrosophic semigroup.

$S_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a, b, c, d \in \langle R \cup I \rangle \right\}$ , neutrosophic semigroup under matrix multiplication and  $S_4 = \langle Z \cup I \rangle$ , neutrosophic semigroup under multiplication.

**Definition 2.13.** Let  $S(N) = \{S_1 \cup S_2 \cup \dots \cup S_N, *_1, \dots, *_N\}$  be a neutrosophic  $N$ -semigroup. A proper subset  $P = \{P_1 \cup P_2 \cup \dots \cup P_N, *_1, *_2, \dots, *_N\}$  of  $S(N)$  is said to be a neutrosophic Nsubsemigroup if  $P_i = P \cap S_i, i = 1, 2, \dots, N$  are subsemigroups of  $S_i$  in which atleast some of the subsemigroups are neutrosophic subsemigroups.

**Definition 2.14.** Let  $S(N) = \{S_1 \cup S_2 \cup \dots \cup S_N, *_1, \dots, *_N\}$  be a neutrosophic strong  $N$ -semigroup. A proper subset  $T = \{T_1 \cup T_2 \cup \dots \cup T_N, *_1, \dots, *_N\}$  of  $S(N)$  is said to be a neutrosophic strong sub  $N$ -semigroup if each  $(T_i, *_i)$  is a neutrosophic subsemigroup of  $(S_i, *_i)$  for  $i = 1, 2, \dots, N$  where  $T_i = T \cap S_i$ .

If only a few of the  $(T_i, *_i)$  in  $T$  are just subsemigroups of  $(S_i, *_i)$  (i.e.  $(T_i, *_i)$  are not neutrosophic subsemigroups then we call  $T$  to be a sub  $N$ -semigroup of  $S(N)$ .

**Definition 2.15.** Let  $S(N) = \{S_1 \cup S_2 \cup \dots \cup S_N, *_1, \dots, *_N\}$  be a neutrosophic  $N$ -semigroup. A proper subset  $P = \{P_1 \cup P_2 \cup \dots \cup P_N, *_1, \dots, *_N\}$  of  $S(N)$  is said to be a neutrosophic  $N$ -subsemigroup, if the following conditions are true,

- i.  $P$  is a neutrosophic sub  $N$ -semigroup of  $S(N)$ .
- ii. Each  $P_i = P \cap S_i, i = 1, 2, \dots, N$  is an ideal of  $S_i$ .

Then  $P$  is called or defined as the neutrosophic  $N$ -ideal of the neutrosophic  $N$ -semigroup  $S(N)$ .

**Definition 2.16.** Let  $S(N) = \{S_1 \cup S_2 \cup \dots \cup S_N, *_1, \dots, *_N\}$  be a neutrosophic strong  $N$ -semigroup. A proper subset  $J = \{I_1 \cup I_2 \cup \dots \cup I_N\}$  where  $I_t = J \cap S_t$  for  $t = 1, 2, \dots, N$  is said to be a neutrosophic strong  $N$ -ideal of  $S(N)$  if the following conditions are satisfied.

1. Each is a neutrosophic subsemigroup of  $S_t, t = 1, 2, \dots, N$  i.e. It is a neutrosophic strong N-subsemigroup of  $S(N)$ .
2. Each is a two sided ideal of  $S_t$  for  $t = 1, 2, \dots, N$ .

Similarly one can define neutrosophic strong  $N$ -left ideal or neutrosophic strong right ideal of  $S(N)$ .

A neutrosophic strong  $N$ -ideal is one which is both a neutrosophic strong  $N$ -left ideal and  $N$ -right ideal of  $S(N)$ .

Throughout this subsection  $U$  refers to an initial universe,  $E$  is a set of parameters,  $P(U)$  is the power set of  $U$ , and  $A \subset E$ . Molodtsov <sup>[12]</sup> defined the soft set in the following manner:

**Definition 2.17.** A pair  $(F, A)$  is called a soft set over  $U$  where  $F$  is a mapping given by  $F : A \longrightarrow P(U)$ .

In other words, a soft set over  $U$  is a parameterized family of subsets of the universe  $U$ . For  $e \in A$ ,  $F(e)$  may be considered as the set of  $e$ -elements of the soft set  $(F, A)$ , or as the set of  $e$ -approximate elements of the soft set.

**Example 2.4.** Suppose that  $U$  is the set of shops.  $E$  is the set of parameters and each parameter is a word or sentence. Let  $E = \{\text{high rent, normal rent, in good condition, in bad condition}\}$ . Let us consider a soft set  $(F, A)$  which describes the attractiveness of shops that Mr.  $Z$  is taking on rent. Suppose that there are five houses in the universe  $U = \{h_1, h_2, h_3, h_4, h_5\}$  under consideration, and that  $A = \{e_1, e_2, e_3\}$  be the set of parameters where

$e_1$  stands for the parameter high rent.

$e_2$  stands for the parameter normal rent.

$e_3$  stands for the parameter in good condition.

Suppose that

$$F(e_1) = \{h_1, h_4\}.$$

$$F(e_2) = \{h_2, h_5\}.$$

$$F(e_3) = \{h_3, h_4, h_5\}.$$

The soft set  $(F, A)$  is an approximated family  $\{F(e_i), i = 1, 2, 3\}$  of subsets of the set  $U$  which gives us a collection of approximate description of an object. Thus, we have the soft set  $(F, A)$  as a collection of approximations as below:

$$(F, A) = \{\text{high rent} = \{h_1, h_4\}, \text{normal rent} = \{h_2, h_5\}, \text{in good condition} = \{h_3, h_4, h_5\}\}.$$

**Definition 2.18.** For two soft sets  $(F, A)$  and  $(H, B)$  over  $U$ ,  $(F, A)$  is called a soft subset of  $(H, B)$  if

1.  $A \subseteq B$ .
2.  $F(e) \subseteq G(e)$ , for all  $e \in A$ .

This relationship is denoted by  $(F, A) \widetilde{\subset} (H, B)$ . Similarly  $(F, A)$  is called a soft superset of  $(H, B)$  if  $(H, B)$  is a soft subset of  $(F, A)$  which is denoted by  $(F, A) \widetilde{\supset} (H, B)$ .

**Definition 2.19.** Two soft sets  $(F, A)$  and  $(H, B)$  over  $U$  are called soft equal if  $(F, A)$  is a soft subset of  $(H, B)$  and  $(H, B)$  is a soft subset of  $(F, A)$ .

**Definition 2.20.**  $(F, A)$  over  $U$  is called an absolute soft set if  $F(e) = U$  for all  $e \in A$  and we denote it by  $U$ .

**Definition 2.21.** Let  $(F, A)$  and  $(G, B)$  be two soft sets over a common universe  $U$  such that  $A \cap B \neq \phi$ . Then their restricted intersection is denoted by  $(F, A) \cap_R (G, B) = (H, C)$  where  $(H, C)$  is defined as  $H(c) = F(c) \cap G(c)$  for all  $c \in C = A \cap B$ .

**Definition 2.22.** The extended intersection of two soft sets  $(F, A)$  and  $(G, B)$  over a common universe  $U$  is the soft set  $(H, C)$ , where  $C = A \cup B$ , and for all  $e \in C$ ,  $H(e)$  is defined as



$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cap G(e), & \text{if } e \in A \cap B. \end{cases}$$

We write  $(F, A) \cap_\varepsilon (G, B) = (H, C)$ .

**Definition 2.23.** The restricted union of two soft sets  $(F, A)$  and  $(G, B)$  over a common universe  $U$  is the soft set  $(H, C)$ , where  $C = A \cup B$ , and for all  $e \in C$ ,  $H(e)$  is defined as the soft set  $(H, C) = (F, A) \cup_R (G, B)$  where  $C = A \cap B$  and  $H(e) = F(e) \cup G(e)$  for all  $e \in C$ .

**Definition 2.24.** The extended union of two soft sets  $(F, A)$  and  $(G, B)$  over a common universe  $U$  is the soft set  $(H, C)$ , where  $C = A \cup B$ , and for all  $e \in C$ ,  $H(e)$  is defined as

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cup G(e), & \text{if } e \in A \cap B. \end{cases}$$

We write  $(F, A) \cup_\varepsilon (G, B) = (H, C)$ .

**Definition 2.25.** A soft set  $(F, A)$  over  $S$  is called a soft semigroup over  $S$  if  $(F, A) \overset{\Delta}{\circ} (F, A) \subseteq (F, A)$ .

It is easy to see that a soft set  $(F, A)$  over  $S$  is a soft semigroup if and only if  $\phi \neq F(a)$  is a subsemigroup of  $S$ .

**Definition 2.26.** A soft set  $(F, A)$  over a semigroup  $S$  is called a soft left (right) ideal over  $S$ , if  $(S, E) \subseteq (F, A)$ ,  $((F, A) \subseteq (S, E))$ .

A soft set over  $S$  is a soft ideal if it is both a soft left and a soft right ideal over  $S$ .

**Proposition 2.1.** A soft set  $(F, A)$  over  $S$  is a soft ideal over  $S$  if and only if  $\phi \neq F(a)$  is an ideal of  $S$ .

**Definition 2.27.** Let  $(G, B)$  be a soft subset of a soft semigroup  $(F, A)$  over  $S$ , then  $(G, B)$  is called a soft subsemigroup (ideal) of  $(F, A)$  if  $G(b)$  is a subsemigroup (ideal) of  $F(b)$  for all  $b \in A$ .

### §3. Soft neutrosophic semigroup

**Definition 3.1.** Let  $N(S)$  be a neutrosophic semigroup and  $(F, A)$  be a soft set over  $N(S)$ . Then  $(F, A)$  is called soft neutrosophic semigroup if and only if  $F(e)$  is neutrosophic subsemigroup of  $N(S)$ , for all  $e \in A$ .

Equivalently  $(F, A)$  is a soft neutrosophic semigroup over  $N(S)$  if  $(F, A) \overset{\Delta}{\circ} (F, A) \subseteq (F, A)$ , where  $\tilde{N}_{(N(S), A)} \neq (F, A) \neq \tilde{\phi}$ .

**Example 3.1.** Let  $N(S) = \langle Z^+ \cup \{0\}^+ \cup \{I\} \rangle$  be a neutrosophic semigroup under  $+$ . Consider  $P = \langle 2Z^+ \cup I \rangle$  and  $R = \langle 3Z^+ \cup I \rangle$  are neutrosophic subsemigroup of  $N(S)$ . Then clearly for all  $e \in A$ ,  $(F, A)$  is a soft neutrosophic semigroup over  $N(S)$ , where  $F(x_1) = \{ \langle 2Z^+ \cup I \rangle \}$ ,  $F(x_2) = \{ \langle 3Z^+ \cup I \rangle \}$ .

**Theorem 3.1.** A soft neutrosophic semigroup over  $N(S)$  always contain a soft semigroup over  $S$ .

**Proof.** The proof of this theorem is straight forward.

**Theorem 3.2.** Let  $(F, A)$  and  $(H, A)$  be two soft neutrosophic semigroups over  $N(S)$ . Then their intersection  $(F, A) \cap (H, A)$  is again soft neutrosophic semigroup over  $N(S)$ .

**Proof.** The proof is straight forward.

**Theorem 3.3.** Let  $(F, A)$  and  $(H, B)$  be two soft neutrosophic semigroups over  $N(S)$ . If  $A \cap B = \phi$ , then  $(F, A) \cup (H, B)$  is a soft neutrosophic semigroup over  $N(S)$ .

**Remark 3.1.** The extended union of two soft neutrosophic semigroups  $(F, A)$  and  $(K, B)$  over  $N(S)$  is not a soft neutrosophic semigroup over  $N(S)$ .

We take the following example for the proof of above remark.

**Example 3.2.** Let  $N(S) = \langle Z^+ \cup I \rangle$  be the neutrosophic semigroup under  $+$ . Take  $P_1 = \{\langle 2Z^+ \cup I \rangle\}$  and  $P_2 = \{\langle 3Z^+ \cup I \rangle\}$  to be any two neutrosophic subsemigroups of  $N(S)$ . Then clearly for all  $e \in A$ ,  $(F, A)$  is a soft neutrosophic semigroup over  $N(S)$ , where  $F(x_1) = \{\langle 2Z^+ \cup I \rangle\}$ ,  $F(x_2) = \{\langle 3Z^+ \cup I \rangle\}$ .

Again Let  $R_1 = \{\langle 5Z^+ \cup I \rangle\}$  and  $R_2 = \{\langle 4Z^+ \cup I \rangle\}$  be another neutrosophic subsemigroups of  $N(S)$  and  $(K, B)$  is another soft neutrosophic semigroup over  $N(S)$ , where  $K(x_1) = \{\langle 5Z^+ \cup I \rangle\}$ ,  $K(x_3) = \{\langle 4Z^+ \cup I \rangle\}$ .

Let  $C = A \cup B$ . The extended union  $(F, A) \cup_\epsilon (K, B) = (H, C)$  where  $x_1 \in C$ , we have  $H(x_1) = F(x_1) \cup K(x_1)$  is not neutrosophic subsemigroup as union of two neutrosophic subsemigroup is not neutrosophic subsemigroup.

**Proposition 3.1.** The extended intersection of two soft neutrosophic semigroups over  $N(S)$  is soft neutrosophic semigroup over  $N(S)$ .

**Remark 3.2.** The restricted union of two soft neutrosophic semigroups  $(F, A)$  and  $(K, B)$  over  $N(S)$  is not a soft neutrosophic semigroup over  $N(S)$ .

We can easily check it in above example.

**Proposition 3.2.** The restricted intersection of two soft neutrosophic semigroups over  $N(S)$  is soft neutrosophic semigroup over  $N(S)$ .

**Proposition 3.3.** The *AND* operation of two soft neutrosophic semigroups over  $N(S)$  is soft neutrosophic semigroup over  $N(S)$ .

**Proposition 3.4.** The *OR* operation of two soft neutrosophic semigroup over  $N(S)$  may not be a soft neutrosophic semigroup over  $N(S)$ .

**Definition 3.2.** Let  $N(S)$  be a neutrosophic monoid and  $(F, A)$  be a soft set over  $N(S)$ . Then  $(F, A)$  is called soft neutrosophic monoid if and only if  $F(e)$  is neutrosophic submonoid of  $N(S)$ , for all  $x \in A$ .

**Example 3.3.** Let  $N(S) = \langle Z \cup I \rangle$  be a neutrosophic monoid under  $+$ . Let  $P = \langle 2Z \cup I \rangle$  and  $Q = \langle 3Z \cup I \rangle$  are neutrosophic submonoids of  $N(S)$ . Then  $(F, A)$  is a soft neutrosophic monoid over  $N(S)$ , where  $F(x_1) = \{\langle 2Z \cup I \rangle\}$ ,  $F(x_2) = \{\langle 3Z \cup I \rangle\}$ .

**Theorem 3.4.** Every soft neutrosophic monoid over  $N(S)$  is a soft neutrosophic semigroup over  $N(S)$  but the converse is not true in general.

**Proof.** The proof is straightforward.

**Proposition 3.5.** Let  $(F, A)$  and  $(K, B)$  be two soft neutrosophic monoids over  $N(S)$ . Then

1. Their extended union  $(F, A) \cup_\varepsilon (K, B)$  over  $N(S)$  is not soft neutrosophic monoid over  $N(S)$ .
2. Their extended intersection  $(F, A) \cap_\varepsilon (K, B)$  over  $N(S)$  is soft neutrosophic monoid over  $N(S)$ .
3. Their restricted union  $(F, A) \cup_R (K, B)$  over  $N(S)$  is not soft neutrosophic monoid over  $N(S)$ .
4. Their restricted intersection  $(F, A) \cap_\varepsilon (K, B)$  over  $N(S)$  is soft neutrosophic monoid over  $N(S)$ .

**Proposition 3.6.** Let  $(F, A)$  and  $(H, B)$  be two soft neutrosophic monoid over  $N(S)$ . Then

1. Their *AND* operation  $(F, A) \wedge (H, B)$  is soft neutrosophic monoid over  $N(S)$ .
2. Their *OR* operation  $(F, A) \vee (H, B)$  is not soft neutrosophic monoid over  $N(S)$ .

**Definition 3.3.** Let  $(F, A)$  be a soft neutrosophic semigroup over  $N(S)$ , then  $(F, A)$  is called Full-soft neutrosophic semigroup over  $N(S)$  if  $F(x) = N(S)$ , for all  $x \in A$ . We denote it by  $N(S)$ .

**Theorem 3.5.** Every Full-soft neutrosophic semigroup over  $N(S)$  always contain absolute soft semigroup over  $S$ .

**Proof.** The proof of this theorem is straight forward.

**Definition 3.4.** Let  $(F, A)$  and  $(H, B)$  be two soft neutrosophic semigroups over  $N(S)$ . Then  $(H, B)$  is a soft neutrosophic subsemigroup of  $(F, A)$ , if

1.  $B \subset A$ .
2.  $H(a)$  is neutrosophic subsemigroup of  $F(a)$ , for all  $a \in B$ .

**Example 3.4.** Let  $N(S) = \langle Z \cup I \rangle$  be a neutrosophic semigroup under  $+$ . Then  $(F, A)$  is a soft neutrosophic semigroup over  $N(S)$ , where  $F(x_1) = \{\langle 2Z \cup I \rangle\}$ ,  $F(x_2) = \{\langle 3Z \cup I \rangle\}$ ,  $F(x_3) = \{\langle 5Z \cup I \rangle\}$ .

Let  $B = \{x_1, x_2\} \subset A$ . Then  $(H, B)$  is soft neutrosophic subsemigroup of  $(F, A)$  over  $N(S)$ , where  $H(x_1) = \{\langle 4Z \cup I \rangle\}$ ,  $H(x_2) = \{\langle 6Z \cup I \rangle\}$ .

**Theorem 3.6.** A soft neutrosophic semigroup over  $N(S)$  have soft neutrosophic subsemigroups as well as soft subsemigroups over  $N(S)$ .

**Proof.** Obvious.

**Theorem 3.7.** Every soft semigroup over  $S$  is always soft neutrosophic subsemigroup of soft neutrosophic semigroup over  $N(S)$ .

**Proof.** The proof is obvious.

**Theorem 3.8.** Let  $(F, A)$  be a soft neutrosophic semigroup over  $N(S)$  and  $\{(H_i, B_i); i \in I\}$  is a non empty family of soft neutrosophic subsemigroups of  $(F, A)$  then

1.  $\cap_{i \in I} (H_i, B_i)$  is a soft neutrosophic subsemigroup of  $(F, A)$ .
2.  $\wedge_{i \in I} (H_i, B_i)$  is a soft neutrosophic subsemigroup of  $\wedge_{i \in I} (F, A)$ .
3.  $\cup_{i \in I} (H_i, B_i)$  is a soft neutrosophic subsemigroup of  $(F, A)$  if  $B_i \cap B_j = \phi$ , for all  $i \neq j$ .

**Proof.** Straightforward.

**Definition 3.5.** A soft set  $(F, A)$  over  $N(S)$  is called soft neutrosophic left (right) ideal over  $N(S)$  if  $N(S) \overset{\wedge}{\circ} (F, A) \subseteq (F, A)$ , where  $\tilde{N}_{(N(S), A)} \neq (F, A) \neq \tilde{\phi}$  and  $N(S)$  is Full-soft neutrosophic semigroup over  $N(S)$ .

A soft set over  $N(S)$  is a soft neutrosophic ideal if it is both a soft neutrosophic left and a soft neutrosophic right ideal over  $N(S)$ .

**Example 3.5.** Let  $N(S) = \langle Z \cup I \rangle$  be the neutrosophic semigroup under multiplication. Let  $P = \langle 2Z \cup I \rangle$  and  $Q = \langle 4Z \cup I \rangle$  are neutrosophic ideals of  $N(S)$ . Then clearly  $(F, A)$  is a soft neutrosophic ideal over  $N(S)$ , where  $F(x_1) = \{\langle 2Z \cup I \rangle\}$ ,  $F(x_2) = \{\langle 4Z \cup I \rangle\}$ .

**Proposition 3.7.**  $(F, A)$  is soft neutrosophic ideal if and only if  $F(x)$  is a neutrosophic ideal of  $N(S)$ , for all  $x \in A$ .

**Theorem 3.9.** Every soft neutrosophic ideal  $(F, A)$  over  $N(S)$  is a soft neutrosophic semigroup but the converse is not true.

**Proposition 3.8.** Let  $(F, A)$  and  $(K, B)$  be two soft neutrosophic ideals over  $N(S)$ . Then

1. Their extended union  $(F, A) \cup_\varepsilon (K, B)$  over  $N(S)$  is soft neutrosophic ideal over  $N(S)$ .
2. Their extended intersection  $(F, A) \cap_\varepsilon (K, B)$  over  $N(S)$  is soft neutrosophic ideal over  $N(S)$ .
3. Their restricted union  $(F, A) \cup_R (K, B)$  over  $N(S)$  is soft neutrosophic ideal over  $N(S)$ .
4. Their restricted intersection  $(F, A) \cap_\varepsilon (K, B)$  over  $N(S)$  is soft neutrosophic ideal over  $N(S)$ .

**Proposition 3.9.**

1. Let  $(F, A)$  and  $(H, B)$  be two soft neutrosophic ideal over  $N(S)$ .
2. Their AND operation  $(F, A) \wedge (H, B)$  is soft neutrosophic ideal over  $N(S)$ .
3. Their OR operation  $(F, A) \vee (H, B)$  is soft neutrosophic ideal over  $N(S)$ .

**Theorem 3.10.** Let  $(F, A)$  and  $(G, B)$  be two soft semigroups (ideals) over  $S$  and  $T$  respectively. Then  $(F, A) \times (G, B)$  is also a soft semigroup (ideal) over  $S \times T$ .

**Proof.** The proof is straight forward.

**Theorem 3.11.** Let  $(F, A)$  be a soft neutrosophic semigroup over  $N(S)$  and  $\{(H_i, B_i); i \in I\}$  is a non empty family of soft neutrosophic ideals of  $(F, A)$  then

1.  $\cap_{i \in I} (H_i, B_i)$  is a soft neutrosophic ideal of  $(F, A)$ .
2.  $\wedge_{i \in I} (H_i, B_i)$  is a soft neutrosophic ideal of  $\wedge_{i \in I} (F, A)$ .
3.  $\cup_{i \in I} (H_i, B_i)$  is a soft neutrosophic ideal of  $(F, A)$ .
4.  $\vee_{i \in I} (H_i, B_i)$  is a soft neutrosophic ideal of  $\vee_{i \in I} (F, A)$ .

**Definition 3.6.** A soft set  $(F, A)$  over  $N(S)$  is called soft neutrosophic principal ideal or soft neutrosophic cyclic ideal if and only if  $F(x)$  is a principal or cyclic neutrosophic ideal of  $N(S)$ , for all  $x \in A$ .

**Proposition 3.10.** Let  $(F, A)$  and  $(K, B)$  be two soft neutrosophic principal ideals over  $N(S)$ . Then

1. Their extended union  $(F, A) \cup_\varepsilon (K, B)$  over  $N(S)$  is not soft neutrosophic principal ideal over  $N(S)$ .
2. Their extended intersection  $(F, A) \cap_\varepsilon (K, B)$  over  $N(S)$  is soft neutrosophic principal ideal over  $N(S)$ .
3. Their restricted union  $(F, A) \cup_R (K, B)$  over  $N(S)$  is not soft neutrosophic principal ideal over  $N(S)$ .

4. Their restricted intersection  $(F, A) \cap_{\varepsilon} (K, B)$  over  $N(S)$  is soft neutrosophic principal ideal over  $N(S)$ .

**Proposition 3.11.** Let  $(F, A)$  and  $(H, B)$  be two soft neutrosophic principal ideals over  $N(S)$ . Then

1. Their *AND* operation  $(F, A) \wedge (H, B)$  is soft neutrosophic principal ideal over  $N(S)$ .
2. Their *OR* operation  $(F, A) \vee (H, B)$  is not soft neutrosophic principal ideal over  $N(S)$ .

### §3. Soft neutrosophic bisemigroup

**Definition 3.1.** Let  $\{BN(S), *_1, *_2\}$  be a neutrosophic bisemigroup and let  $(F, A)$  be a soft set over  $BN(S)$ . Then  $(F, A)$  is said to be soft neutrosophic bisemigroup over  $BN(S)$  if and only if  $F(x)$  is neutrosophic subbisemigroup of  $BN(S)$  for all  $x \in A$ .

**Example 3.1.** Let  $BN(S) = \{0, 1, 2, I, 2I, \langle Z \cup I \rangle, \times, +\}$  be a neutrosophic bisemigroup. Let  $T = \{0, I, 2I, \langle 2Z \cup I \rangle, \times, +\}$ ,  $P = \{0, 1, 2, \langle 5Z \cup I \rangle, \times, +\}$  and  $L = \{0, 1, 2, Z, \times, +\}$  are neutrosophic subbisemigroup of  $BN(S)$ . The  $(F, A)$  is clearly soft neutrosophic bisemigroup over  $BN(S)$ , where  $F(x_1) = \{0, I, 2I, \langle 2Z \cup I \rangle, \times, +\}$ ,  $F(x_2) = \{0, 1, 2, \langle 5Z \cup I \rangle, \times, +\}$ ,  $F(x_3) = \{0, 1, 2, Z, \times, +\}$ .

**Theorem 3.1.** Let  $(F, A)$  and  $(H, A)$  be two soft neutrosophic bisemigroup over  $BN(S)$ . Then their intersection  $(F, A) \cap (H, A)$  is again a soft neutrosophic bisemigroup over  $BN(S)$ .

**Proof.** Straightforward.

**Theorem 3.2.** Let  $(F, A)$  and  $(H, B)$  be two soft neutrosophic bisemigroups over  $BN(S)$  such that  $A \cap B = \phi$ , then their union is soft neutrosophic bisemigroup over  $BN(S)$ .

**Proof.** Straightforward.

**Proposition 3.1.** Let  $(F, A)$  and  $(K, B)$  be two soft neutrosophic bisemigroups over  $BN(S)$ . Then

1. Their extended union  $(F, A) \cup_{\varepsilon} (K, B)$  over  $BN(S)$  is not soft neutrosophic bisemigroup over  $BN(S)$ .
2. Their extended intersection  $(F, A) \cap_{\varepsilon} (K, B)$  over  $BN(S)$  is soft neutrosophic bisemigroup over  $BN(S)$ .
3. Their restricted union  $(F, A) \cup_R (K, B)$  over  $BN(S)$  is not soft neutrosophic bisemigroup over  $BN(S)$ .
4. Their restricted intersection  $(F, A) \cap_{\varepsilon} (K, B)$  over  $BN(S)$  is soft neutrosophic bisemigroup over  $BN(S)$ .

**Proposition 3.2.** Let  $(F, A)$  and  $(K, B)$  be two soft neutrosophic bisemigroups over  $BN(S)$ . Then

1. Their *AND* operation  $(F, A) \wedge (K, B)$  is soft neutrosophic bisemigroup over  $BN(S)$ .
2. Their *OR* operation  $(F, A) \vee (K, B)$  is not soft neutrosophic bisemigroup over  $BN(S)$ .

**Definition 3.2.** Let  $(F, A)$  be a soft neutrosophic bisemigroup over  $BN(S)$ , then  $(F, A)$  is called Full-soft neutrosophic bisemigroup over  $BN(S)$  if  $F(x) = BN(S)$ , for all  $x \in A$ . We denote it by  $BN(S)$ .

**Definition 3.3.** Let  $(F, A)$  and  $(H, B)$  be two soft neutrosophic bisemigroups over  $BN(S)$ . Then  $(H, B)$  is a soft neutrosophic subbisemigroup of  $(F, A)$ , if

1.  $B \subset A$ .
2.  $H(x)$  is neutrosophic subbisemigroup of  $F(x)$ , for all  $x \in B$ .

**Example 3.2.** Let  $BN(S) = \{0, 1, 2, I, 2I, \langle Z \cup I \rangle, \times, +\}$  be a neutrosophic bisemigroup. Let  $T = \{0, I, 2I, \langle 2Z \cup I \rangle, \times, +\}$ ,  $P = \{0, 1, 2, \langle 5Z \cup I \rangle, \times, +\}$  and  $L = \{0, 1, 2, Z, \times, +\}$  are neutrosophic subbisemigroup of  $BN(S)$ . The  $(F, A)$  is clearly soft neutrosophic bisemigroup over  $BN(S)$ , where  $F(x_1) = \{0, I, 2I, \langle 2Z \cup I \rangle, \times, +\}$ ,  $F(x_2) = \{0, 1, 2, \langle 5Z \cup I \rangle, \times, +\}$ ,  $F(x_3) = \{0, 1, 2, Z, \times, +\}$ .

Then  $(H, B)$  is a soft neutrosophic subbisemigroup of  $(F, A)$ , where  $H(x_1) = \{0, I, \langle 4Z \cup I \rangle, \times, +\}$ ,  $H(x_3) = \{0, 1, 4Z, \times, +\}$ .

**Theorem 3.3.** Let  $(F, A)$  be a soft neutrosophic bisemigroup over  $BN(S)$  and  $\{(H_i, B_i); i \in I\}$  be a non-empty family of soft neutrosophic subbisemigroups of  $(F, A)$  then

1.  $\cap_{i \in I} (H_i, B_i)$  is a soft neutrosophic subbisemigroup of  $(F, A)$ .
2.  $\wedge_{i \in I} (H_i, B_i)$  is a soft neutrosophic subbisemigroup of  $\wedge_{i \in I} (F, A)$ .
3.  $\cup_{i \in I} (H_i, B_i)$  is a soft neutrosophic subbisemigroup of  $(F, A)$  if  $B_i \cap B_j = \phi$ , for all  $i \neq j$ .

**Proof.** Straightforward.

**Theorem 3.4.**  $(F, A)$  is called soft neutrosophic biideal over  $BN(S)$  if  $F(x)$  is neutrosophic biideal of  $BN(S)$ , for all  $x \in A$ .

**Example 3.3.** Let  $BN(S) = (\{ \langle Z \cup I \rangle, 0, 1, 2, I, 2I, +, \times \} (\times \text{ under multiplication modulo } 3))$ . Let  $T = \{ \langle 2Z \cup I \rangle, 0, I, 1, 2I, +, \times \}$  and  $J = \{ \langle 8Z \cup I \rangle, \{0, 1, I, 2I\}, +, \times \}$  are ideals of  $BN(S)$ . Then  $(F, A)$  is soft neutrosophic biideal over  $BN(S)$ , where  $F(x_1) = \{ \langle 2Z \cup I \rangle, 0, I, 1, 2I, +, \times \}$ ,  $F(x_2) = \{ \langle 8Z \cup I \rangle, \{0, 1, I, 2I\}, +, \times \}$ .

**Theorem 3.5.** Every soft neutrosophic biideal  $(F, A)$  over  $BS(N)$  is a soft neutrosophic bisemigroup but the converse is not true.

**Proposition 3.3.** Let  $(F, A)$  and  $(K, B)$  be two soft neutrosophic biideals over  $BN(S)$ . Then

1. Their extended union  $(F, A) \cup_\varepsilon (K, B)$  over  $BN(S)$  is not soft neutrosophic biideal over  $BN(S)$ .
2. Their extended intersection  $(F, A) \cap_\varepsilon (K, B)$  over  $BN(S)$  is soft neutrosophic biideal over  $BN(S)$ .
3. Their restricted union  $(F, A) \cup_R (K, B)$  over  $BN(S)$  is not soft neutrosophic biideal over  $BN(S)$ .
4. Their restricted intersection  $(F, A) \cap_\varepsilon (K, B)$  over  $BN(S)$  is soft neutrosophic biideal over  $BN(S)$ .

**Proposition 3.4.** Let  $(F, A)$  and  $(H, B)$  be two soft neutrosophic biideal over  $BN(S)$ . Then

1. Their AND operation  $(F, A) \wedge (H, B)$  is soft neutrosophic biideal over  $BN(S)$ .
2. Their OR operation  $(F, A) \vee (H, B)$  is not soft neutrosophic biideal over  $BN(S)$ .

**Theorem 3.6.**

Let  $(F, A)$  be a soft neutrosophic bisemigroup over  $BN(S)$  and  $\{(H_i, B_i); i \in I\}$  is a non empty family of soft neutrosophic biideals of  $(F, A)$  then

1.  $\cap_{i \in I} (H_i, B_i)$  is a soft neutrosophic biideal of  $(F, A)$ .
2.  $\wedge_{i \in I} (H_i, B_i)$  is a soft neutrosophic biideal of  $\wedge_{i \in I} (F, A)$ .

## §4. Soft neutrosophic strong bisemigroup

**Definition 4.1.** Let  $(F, A)$  be a soft set over a neutrosophic bisemigroup  $BN(S)$ . Then  $(F, A)$  is said to be soft strong neutrosophic bisemigroup over  $BN(G)$  if and only if  $F(x)$  is neutrosophic strong subbisemigroup of  $BN(G)$  for all  $x \in A$ .

**Example 4.1.** Let  $BN(S) = \{0, 1, 2, I, 2I, \langle Z \cup I \rangle, \times, +\}$  be a neutrosophic bisemigroup. Let  $T = \{0, I, 2I, \langle 2Z \cup I \rangle, \times, +\}$  and  $R = \{0, 1, I, \langle 4Z \cup I \rangle, \times, +\}$  are neutrosophic strong subbisemigroups of  $BN(S)$ . Then  $(F, A)$  is soft neutrosophic strong bisemigroup over  $BN(S)$ , where  $F(x_1) = \{0, I, 2I, \langle 2Z \cup I \rangle, \times, +\}$ ,  $F(x_2) = \{0, I, 1, \langle 4Z \cup I \rangle, \times, +\}$ .

**Theorem 4.1.** Every soft neutrosophic strong bisemigroup is a soft neutrosophic bisemigroup but the converse is not true.

**Proposition 4.1.** Let  $(F, A)$  and  $(K, B)$  be two soft neutrosophic strong bisemigroups over  $BN(S)$ . Then

1. Their extended union  $(F, A) \cup_{\varepsilon} (K, B)$  over  $BN(S)$  is not soft neutrosophic strong bisemigroup over  $BN(S)$ .
2. Their extended intersection  $(F, A) \cap_{\varepsilon} (K, B)$  over  $BN(S)$  is soft neutrosophic strong bisemigroup over  $BN(S)$ .
3. Their restricted union  $(F, A) \cup_R (K, B)$  over  $BN(S)$  is not soft neutrosophic strong bisemigroup over  $BN(S)$ .
4. Their restricted intersection  $(F, A) \cap_{\varepsilon} (K, B)$  over  $BN(S)$  is soft neutrosophic strong bisemigroup over  $BN(S)$ .

**Proposition 4.2.** Let  $(F, A)$  and  $(K, B)$  be two soft neutrosophic strong bisemigroups over  $BN(S)$ . Then

1. Their AND operation  $(F, A) \wedge (K, B)$  is soft neutrosophic strong bisemigroup over  $BN(S)$ .
2. Their OR operation  $(F, A) \vee (K, B)$  is not soft neutrosophic strong bisemigroup over  $BN(S)$ .

**Definition 4.2.** Let  $(F, A)$  and  $(H, B)$  be two soft neutrosophic strong bisemigroups over  $BN(S)$ . Then  $(H, B)$  is a soft neutrosophic strong subbisemigroup of  $(F, A)$ , if

1.  $B \subset A$ .
2.  $H(x)$  is neutrosophic strong subbisemigroup of  $F(x)$ , for all  $x \in B$ .

**Example 4.2.** Let  $BN(S) = \{0, 1, 2, I, 2I, \langle Z \cup I \rangle, \times, +\}$  be a neutrosophic bisemigroup. Let  $T = \{0, I, 2I, \langle 2Z \cup I \rangle, \times, +\}$  and  $R = \{0, 1, I, \langle 4Z \cup I \rangle, \times, +\}$  are neutrosophic strong subbisemigroups of  $BN(S)$ . Then  $(F, A)$  is soft neutrosophic strong bisemigroup over  $BN(S)$ , where  $F(x_1) = \{0, I, 2I, \langle 2Z \cup I \rangle, \times, +\}$ ,  $F(x_2) = \{0, I, \langle 4Z \cup I \rangle, \times, +\}$ .

Then  $(H, B)$  is a soft neutrosophic strong subbisemigroup of  $(F, A)$ , where  $H(x_1) = \{0, I, \langle 4Z \cup I \rangle, \times, +\}$ .

**Theorem 4.2.** Let  $(F, A)$  be a soft neutrosophic strong bisemigroup over  $BN(S)$  and  $\{(H_i, B_i); i \in I\}$  be a non empty family of soft neutrosophic strong subbisemigroups of  $(F, A)$  then

1.  $\cap_{i \in I} (H_i, B_i)$  is a soft neutrosophic strong subbisemigroup of  $(F, A)$ .
2.  $\wedge_{i \in I} (H_i, B_i)$  is a soft neutrosophic strong subbisemigroup of  $\wedge_{i \in I} (F, A)$ .

3.  $\cup_{i \in I} (H_i, B_i)$  is a soft neutrosophic strong subbisemigroup of  $(F, A)$  if  $B_i \cap B_j = \phi$ , for all  $i \neq j$ .

**Proof.** Straightforward.

**Definition 4.3.**  $(F, A)$  over  $BN(S)$  is called soft neutrosophic strong biideal if  $F(x)$  is neutrosophic strong biideal of  $BN(S)$ , for all  $x \in A$ .

**Example 4.3.** Let  $BN(S) = (\{ \langle Z \cup I \rangle, 0, 1, 2, I, 2I \}, +, \times (\times \text{ under multiplication modulo } 3))$ . Let  $T = \{ \langle 2Z \cup I \rangle, 0, I, 1, 2I, +, \times \}$  and  $J = \{ \langle 8Z \cup I \rangle, \{0, 1, I, 2I\}, +, \times \}$  are neutrosophic strong ideals of  $BN(S)$ . Then  $(F, A)$  is soft neutrosophic strong biideal over  $BN(S)$ , where  $F(x_1) = \{ \langle 2Z \cup I \rangle, 0, I, 1, 2I, +, \times \}$ ,  $F(x_2) = \{ \langle 8Z \cup I \rangle, \{0, 1, I, 2I\}, +, \times \}$ .

**Theorem 4.3.** Every soft neutrosophic strong biideal  $(F, A)$  over  $BS(N)$  is a soft neutrosophic bisemigroup but the converse is not true.

**Theorem 4.4.** Every soft neutrosophic strong biideal  $(F, A)$  over  $BS(N)$  is a soft neutrosophic strong bisemigroup but the converse is not true.

**Proposition 4.3.** Let  $(F, A)$  and  $(K, B)$  be two soft neutrosophic strong biideals over  $BN(S)$ . Then

1. Their extended union  $(F, A) \cup_\epsilon (K, B)$  over  $BN(S)$  is not soft neutrosophic strong biideal over  $BN(S)$ .
2. Their extended intersection  $(F, A) \cap_\epsilon (K, B)$  over  $BN(S)$  is soft neutrosophic strong biideal over  $BN(S)$ .
3. Their restricted union  $(F, A) \cup_R (K, B)$  over  $BN(S)$  is not soft neutrosophic strong biideal over  $BN(S)$ .
4. Their restricted intersection  $(F, A) \cap_\epsilon (K, B)$  over  $BN(S)$  is soft neutrosophic strong biideal over  $BN(S)$ .

**Proposition 4.4.** Let  $(F, A)$  and  $(H, B)$  be two soft neutrosophic strong biideal over  $BN(S)$ . Then

1. Their AND operation  $(F, A) \wedge (H, B)$  is soft neutrosophic strong biideal over  $BN(S)$ .
2. Their OR operation  $(F, A) \vee (H, B)$  is not soft neutrosophic strong biideal over  $BN(S)$ .

**Theorem 4.5.** Let  $(F, A)$  be a soft neutrosophic strong bisemigroup over  $BN(S)$  and  $\{(H_i, B_i); i \in I\}$  is a non empty family of soft neutrosophic strong biideals of  $(F, A)$  then

1.  $\cap_{i \in I} (H_i, B_i)$  is a soft neutrosophic strong biideal of  $(F, A)$ .
2.  $\wedge_{i \in I} (H_i, B_i)$  is a soft neutrosophic strong biideal of  $\wedge_{i \in I} (F, A)$ .

## §5. Soft neutrosophic $N$ -semigroup

**Definition 5.1.** Let  $\{S(N), *_1, \dots, *_N\}$  be a neutrosophic  $N$ -semigroup and  $(F, A)$  be a soft set over  $\{S(N), *_1, \dots, *_N\}$ . Then  $(F, A)$  is termed as soft neutrosophic  $N$ -semigroup if and only if  $F(x)$  is neutrosophic sub  $N$ -semigroup, for all  $x \in A$ .

**Example 5.1.** Let  $S(N) = \{S_1 \cup S_2 \cup S_3 \cup S_4, *_1, *_2, *_3, *_4\}$  be a neutrosophic 4-semigroup where

$S_1 = \{Z_{12}, \text{ semigroup under multiplication modulo } 12\}$ .

$S_2 = \{0, 1, 2, 3, I, 2I, 3I, \text{ semigroup under multiplication modulo } 4\}$ , a neutrosophic semigroup.



$S_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a, b, c, d \in \langle R \cup I \rangle \right\}$ , neutrosophic semigroup under matrix multiplication.

$S_4 = \langle Z \cup I \rangle$ , neutrosophic semigroup under multiplication. Let  $T = \{T_1 \cup T_2 \cup T_3 \cup T_4, *_1, *_2, *_3, *_4\}$  is a neutrosophic sub 4-semigroup of  $S(4)$ , where  $T_1 = \{0, 2, 4, 6, 8, 10\} \subseteq Z_{12}$ ,  $T_2 = \{0, I, 2I, 3I\} \subset S_2$ ,  $T_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a, b, c, d \in \langle Q \cup I \rangle \right\} \subset S_3$ ,  $T_4 = \{\langle 5Z \cup I \rangle\} \subset S_4$ , the neutrosophic semigroup under multiplication. Also let  $P = \{P_1 \cup P_2 \cup P_3 \cup P_4, *_1, *_2, *_3, *_4\}$  be another neutrosophic sub 4-semigroup of  $S(4)$ , where  $P_1 = \{0, 6\} \subseteq Z_{12}$ ,  $P_2 = \{0, 1, I\} \subset S_2$ ,  $P_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a, b, c, d \in \langle Z \cup I \rangle \right\} \subset S_3$ ,  $P_4 = \{\langle 2Z \cup I \rangle\} \subset S_4$ . Then  $(F, A)$  is soft neutrosophic 4-semigroup over  $S(4)$ , where

$$F(x_1) = \{0, 2, 4, 6, 8, 10\} \cup \{0, I, 2I, 3I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a, b, c, d \in \langle Q \cup I \rangle \right\} \cup \{\langle 5Z \cup I \rangle\},$$

$$F(x_2) = \{0, 6\} \cup \{0, 1, I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a, b, c, d \in \langle Z \cup I \rangle \right\} \cup \{\langle 2Z \cup I \rangle\}.$$

**Theorem 5.1.** Let  $(F, A)$  and  $(H, A)$  be two soft neutrosophic  $N$ -semigroup over  $S(N)$ . Then their intersection  $(F, A) \cap (H, A)$  is again a soft neutrosophic  $N$ -semigroup over  $S(N)$ .

**Proof.** Straightforward.

**Theorem 5.2.** Let  $(F, A)$  and  $(H, B)$  be two soft neutrosophic  $N$ -semigroups over  $S(N)$  such that  $A \cap B = \phi$ , then their union is soft neutrosophic  $N$ -semigroup over  $S(N)$ .

**Proof.** Straightforward.

**Proposition 5.1.** Let  $(F, A)$  and  $(K, B)$  be two soft neutrosophic  $N$ -semigroups over  $S(N)$ . Then

1. Their extended union  $(F, A) \cup_\varepsilon (K, B)$  over  $S(N)$  is not soft neutrosophic  $N$ -semigroup over  $S(N)$ .
2. Their extended intersection  $(F, A) \cap_\varepsilon (K, B)$  over  $S(N)$  is soft neutrosophic  $N$ -semigroup over  $S(N)$ .
3. Their restricted union  $(F, A) \cup_R (K, B)$  over  $S(N)$  is not soft neutrosophic  $N$ -semigroup over  $S(N)$ .
4. Their restricted intersection  $(F, A) \cap_\varepsilon (K, B)$  over  $S(N)$  is soft neutrosophic  $N$ -semigroup over  $S(N)$ .

**Proposition 5.2.** Let  $(F, A)$  and  $(K, B)$  be two soft neutrosophic  $N$ -semigroups over  $S(N)$ . Then

1. Their AND operation  $(F, A) \wedge (K, B)$  is soft neutrosophic  $N$ -semigroup over  $S(N)$ .
2. Their OR operation  $(F, A) \vee (K, B)$  is not soft neutrosophic  $N$ -semigroup over  $S(N)$ .

**Definition 5.2.** Let  $(F, A)$  be a soft neutrosophic  $N$ -semigroup over  $S(N)$ , then  $(F, A)$  is called Full-soft neutrosophic  $N$ -semigroup over  $S(N)$  if  $F(x) = S(N)$ , for all  $x \in A$ . We denote it by  $S(N)$ .

**Definition 5.3.** Let  $(F, A)$  and  $(H, B)$  be two soft neutrosophic  $N$ -semigroups over  $S(N)$ . Then  $(H, B)$  is a soft neutrosophic sub  $N$ -semigroup of  $(F, A)$ , if

1.  $B \subset A$ .
2.  $H(x)$  is neutrosophic sub  $N$ -semigroup of  $F(x)$ , for all  $x \in B$ .

**Example 5.2.** Let  $S(N) = \{S_1 \cup S_2 \cup S_3 \cup S_4, *_1, *_2, *_3, *_4\}$  be a neutrosophic 4-semigroup where

$S_1 = \{Z_{12}, \text{semigroup under multiplication modulo } 12\}$ .

$S_2 = \{0, 1, 2, 3, I, 2I, 3I, \text{semigroup under multiplication modulo } 4\}$ , a neutrosophic semigroup.

$S_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle R \cup I \rangle \right\}$ , neutrosophic semigroup under matrix multiplication.

$S_4 = \langle Z \cup I \rangle$ , neutrosophic semigroup under multiplication. Let  $T = \{T_1 \cup T_2 \cup T_3 \cup T_4, *_1, *_2, *_3, *_4\}$  is a neutrosophic sub 4-semigroup of  $S(4)$ , where  $T_1 = \{0, 2, 4, 6, 8, 10\} \subseteq Z_{12}$ ,  $T_2 = \{0, I, 2I, 3I\} \subset S_2$ ,  $T_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Q \cup I \rangle \right\} \subset S_3$ ,  $T_4 = \{\langle 5Z \cup I \rangle\} \subset S_4$ , the neutrosophic semigroup under multiplication. Also let  $P = \{P_1 \cup P_2 \cup P_3 \cup P_4, *_1, *_2, *_3, *_4\}$  be another neutrosophic sub 4-semigroup of  $S(4)$ , where  $P_1 = \{0, 6\} \subseteq Z_{12}$ ,  $P_2 = \{0, 1, I\} \subset S_2$ ,  $P_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Z \cup I \rangle \right\} \subset S_3$ ,  $P_4 = \{\langle 2Z \cup I \rangle\} \subset S_4$ . Also let  $R = \{R_1 \cup R_2 \cup R_3 \cup R_4, *_1, *_2, *_3, *_4\}$  be a neutrosophic sub 4-semigroup of  $S(4)$  where  $R_1 = \{0, 3, 6, 9\}$ ,  $R_2 = \{0, I, 2I\}$ ,  $R_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle 2Z \cup I \rangle \right\}$ ,  $R_4 = \{\langle 3Z \cup I \rangle\}$ . Then  $(F, A)$  is soft neutrosophic 4-semigroup over  $S(4)$ , where

$$F(x_1) = \{0, 2, 4, 6, 8, 10\} \cup \{0, I, 2I, 3I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Q \cup I \rangle \right\} \cup \{\langle 5Z \cup I \rangle\},$$

$$F(x_2) = \{0, 6\} \cup \{0, 1, I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Z \cup I \rangle \right\} \cup \{\langle 2Z \cup I \rangle\},$$

$$F(x_3) = \{0, 3, 6, 9\} \cup \{0, I, 2I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle 2Z \cup I \rangle \right\} \cup \{\langle 3Z \cup I \rangle\}.$$

Clearly  $(H, B)$  is a soft neutrosophic sub  $N$ -semigroup of  $(F, A)$ , where

$$H(x_1) = \{0, 4, 8\} \cup \{0, I, 2I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Z \cup I \rangle \right\} \cup \{\langle 10Z \cup I \rangle\},$$

$$H(x_3) = \{0, 6\} \cup \{0, I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle 4Z \cup I \rangle \right\} \cup \{\langle 6Z \cup I \rangle\}.$$

**Theorem 5.3.** Let  $(F, A)$  be a soft neutrosophic  $N$ -semigroup over  $S(N)$  and  $\{(H_i, B_i); i \in I\}$  is a non empty family of soft neutrosophic sub  $N$ -semigroups of  $(F, A)$  then

1.  $\cap_{i \in I} (H_i, B_i)$  is a soft neutrosophic sub  $N$ -semigroup of  $(F, A)$ .
2.  $\wedge_{i \in I} (H_i, B_i)$  is a soft neutrosophic sub  $N$ -semigroup of  $\wedge_{i \in I} (F, A)$ .
3.  $\cup_{i \in I} (H_i, B_i)$  is a soft neutrosophic sub  $N$ -semigroup of  $(F, A)$  if  $B_i \cap B_j = \phi$ , for all  $i \neq j$ .

**Proof.** Straightforward.

**Definition 5.4.**  $(F, A)$  over  $S(N)$  is called soft neutrosophic  $N$ -ideal if  $F(x)$  is neutrosophic  $N$ -ideal of  $S(N)$ , for all  $x \in A$ .

**Theorem 5.4.** Every soft neutrosophic  $N$ -ideal  $(F, A)$  over  $S(N)$  is a soft neutrosophic  $N$ -semigroup but the converse is not true.

**Proposition 5.3.** Let  $(F, A)$  and  $(K, B)$  be two soft neutrosophic  $N$ -ideals over  $S(N)$ . Then

1. Their extended union  $(F, A) \cup_\varepsilon (K, B)$  over  $S(N)$  is not soft neutrosophic  $N$ -ideal over  $S(N)$ .
2. Their extended intersection  $(F, A) \cap_\varepsilon (K, B)$  over  $S(N)$  is soft neutrosophic  $N$ -ideal over  $S(N)$ .
3. Their restricted union  $(F, A) \cup_R (K, B)$  over  $S(N)$  is not soft neutrosophic  $N$ -ideal over  $S(N)$ .
4. Their restricted intersection  $(F, A) \cap_\varepsilon (K, B)$  over  $S(N)$  is soft neutrosophic  $N$ -ideal over  $S(N)$ .

**Proposition 5.4.** Let  $(F, A)$  and  $(H, B)$  be two soft neutrosophic  $N$ -ideal over  $S(N)$ . Then

1. Their AND operation  $(F, A) \wedge (H, B)$  is soft neutrosophic  $N$ -ideal over  $S(N)$ .
2. Their OR operation  $(F, A) \vee (H, B)$  is not soft neutrosophic  $N$ -ideal over  $S(N)$ .

**Theorem 5.5.** Let  $(F, A)$  be a soft neutrosophic  $N$ -semigroup over  $S(N)$  and  $\{(H_i, B_i); i \in I\}$  is a non empty family of soft neutrosophic  $N$ -ideals of  $(F, A)$  then

1.  $\cap_{i \in I} (H_i, B_i)$  is a soft neutrosophic  $N$ -ideal of  $(F, A)$ .
2.  $\wedge_{i \in I} (H_i, B_i)$  is a soft neutrosophic  $N$ -ideal of  $\wedge_{i \in I} (F, A)$ .

## §6. Soft neutrosophic strong $N$ -semigroup

**Definition 6.1.** Let  $\{S(N), *_1, \dots, *_N\}$  be a neutrosophic  $N$ -semigroup and  $(F, A)$  be a soft set over  $\{S(N), *_1, \dots, *_N\}$ . Then  $(F, A)$  is called soft neutrosophic strong  $N$ -semigroup if and only if  $F(x)$  is neutrosophic strong sub  $N$ -semigroup, for all  $x \in A$ .

**Example 6.1.** Let  $S(N) = \{S_1 \cup S_2 \cup S_3 \cup S_4, *_1, *_2, *_3, *_4\}$  be a neutrosophic 4-semigroup where

$S_1 = \langle Z_6 \cup I \rangle$ , a neutrosophic semigroup.

$S_2 = \{0, 1, 2, 3, I, 2I, 3I\}$ , semigroup under multiplication modulo 4, a neutrosophic semigroup.

$S_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle R \cup I \rangle \right\}$ , neutrosophic semigroup under matrix multiplication.

$S_4 = \langle Z \cup I \rangle$ , neutrosophic semigroup under multiplication. Let  $T = \{T_1 \cup T_2 \cup T_3 \cup T_4, *_1, *_2, *_3, *_4\}$  is a neutrosophic strong sub 4-semigroup of  $S(4)$ , where  $T_1 = \{0, 3, 3I\} \subseteq \langle Z_6 \cup I \rangle$ ,  $T_2 = \{0, I, 2I, 3I\} \subset S_2$ ,  $T_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Q \cup I \rangle \right\} \subset S_3$ ,  $T_4 = \{\langle 5Z \cup I \rangle\} \subset S_4$ , the neutrosophic semigroup under multiplication. Also let  $P = \{P_1 \cup P_2 \cup P_3 \cup P_4, *_1, *_2, *_3, *_4\}$  be another neutrosophic strong sub 4-semigroup of  $S(4)$ , where  $P_1 = \{0, 2I, 4I\} \subseteq \langle Z_6 \cup I \rangle$ ,  $P_2 = \{0, 1, I\} \subset S_2$ ,  $P_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Z \cup I \rangle \right\} \subset S_3$ ,  $P_4 = \{\langle 2Z \cup I \rangle\} \subset S_4$ . Then  $(F, A)$  is soft neutrosophic strong 4-semigroup over  $S(4)$ , where Then  $(F, A)$  is soft neutrosophic 4-semigroup over  $S(4)$ , where

$$F(x_1) = \{0, 3, 3I\} \cup \{0, I, 2I, 3I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Q \cup I \rangle \right\} \cup \{\langle 5Z \cup I \rangle\},$$

$$F(x_2) = \{0, 2I, 4I\} \cup \{0, 1, I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Z \cup I \rangle \right\} \cup \{\langle 2Z \cup I \rangle\}.$$

**Theorem 6.1.** Every soft neutrosophic strong  $N$ -semigroup is trivially a soft neutrosophic  $N$ -semigroup but the converse is not true.

**Proposition 6.1.** Let  $(F, A)$  and  $(K, B)$  be two soft neutrosophic strong  $N$ -semigroups over  $S(N)$ . Then

1. Their extended union  $(F, A) \cup_\varepsilon (K, B)$  over  $S(N)$  is not soft neutrosophic strong  $N$ -semigroup over  $S(N)$ .
2. Their extended intersection  $(F, A) \cap_\varepsilon (K, B)$  over  $S(N)$  is soft neutrosophic strong  $N$ -semigroup over  $S(N)$ .
3. Their restricted union  $(F, A) \cup_R (K, B)$  over  $S(N)$  is not soft neutrosophic strong  $N$ -semigroup over  $S(N)$ .
4. Their restricted intersection  $(F, A) \cap_\varepsilon (K, B)$  over  $S(N)$  is soft neutrosophic strong  $N$ -semigroup over  $S(N)$ .

**Proposition 6.2.** Let  $(F, A)$  and  $(K, B)$  be two soft neutrosophic strong  $N$ -semigroups over  $S(N)$ . Then

1. Their *AND* operation  $(F, A) \wedge (K, B)$  is soft neutrosophic strong  $N$ -semigroup over  $S(N)$ .
2. Their *OR* operation  $(F, A) \vee (K, B)$  is not soft neutrosophic strong  $N$ -semigroup over  $S(N)$ .

**Definition 6.2.** Let  $(F, A)$  and  $(H, B)$  be two soft neutrosophic strong  $N$ -semigroups over  $S(N)$ . Then  $(H, B)$  is a soft neutrosophic strong sub  $N$ -semigroup of  $(F, A)$ , if

1.  $B \subset A$ .
2.  $H(x)$  is neutrosophic strong sub  $N$ -semigroup of  $F(x)$ , for all  $x \in B$ .

**Theorem 6.2.**

1. Let  $(F, A)$  be a soft neutrosophic strong  $N$ -semigroup over  $S(N)$  and  $\{(H_i, B_i); i \in I\}$  is a non empty family of soft neutrosophic strong sub  $N$ -semigroups of  $(F, A)$  then
2.  $\cap_{i \in I} (H_i, B_i)$  is a soft neutrosophic strong sub  $N$ -semigroup of  $(F, A)$ .

3.  $\wedge_{i \in I} (H_i, B_i)$  is a soft neutrosophic strong sub  $N$ -semigroup of  $\wedge_{i \in I} (F, A)$ .
4.  $\cup_{i \in I} (H_i, B_i)$  is a soft neutrosophic strong sub  $N$ -semigroup of  $(F, A)$  if  $B_i \cap B_j = \phi$ , for all  $i \neq j$ .

**Proof.** Straightforward.

**Definition 6.3.**  $(F, A)$  over  $S(N)$  is called soft neutrosophic strong  $N$ -ideal if  $F(x)$  is neutrosophic strong  $N$ -ideal of  $S(N)$ , for all  $x \in A$ .

**Theorem 6.3.** Every soft neutrosophic strong  $N$ -ideal  $(F, A)$  over  $S(N)$  is a soft neutrosophic strong  $N$ -semigroup but the converse is not true.

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1. Their extended union  $(F, A) \cup_\varepsilon (K, B)$  over  $S(N)$  is not soft neutrosophic strong  $N$ -ideal over  $S(N)$ .
2. Their extended intersection  $(F, A) \cap_\varepsilon (K, B)$  over  $S(N)$  is soft neutrosophic strong  $N$ -ideal over  $S(N)$ .
3. Their restricted union  $(F, A) \cup_R (K, B)$  over  $S(N)$  is not soft neutrosophic strong  $N$ -ideal over  $S(N)$ .
4. Their restricted intersection  $(F, A) \cap_\varepsilon (K, B)$  over  $S(N)$  is soft neutrosophic strong  $N$ -ideal over  $S(N)$ .

**Proposition 6.4.** Let  $(F, A)$  and  $(H, B)$  be two soft neutrosophic strong  $N$ -ideal over  $S(N)$ . Then

1. Their *AND* operation  $(F, A) \wedge (H, B)$  is soft neutrosophic strong  $N$ -ideal over  $S(N)$ .
2. Their *OR* operation  $(F, A) \vee (H, B)$  is not soft neutrosophic strong  $N$ -ideal over  $S(N)$ .

**Theorem 6.5.** Let  $(F, A)$  be a soft neutrosophic strong  $N$ -semigroup over  $S(N)$  and  $\{(H_i, B_i); i \in I\}$  is a non empty family of soft neutrosophic strong  $N$ -ideals of  $(F, A)$  then

1.  $\cap_{i \in I} (H_i, B_i)$  is a soft neutrosophic strong  $N$ -ideal of  $(F, A)$ .
2.  $\wedge_{i \in I} (H_i, B_i)$  is a soft neutrosophic strong  $N$ -ideal of  $\wedge_{i \in I} (F, A)$ .

## Conclusion

This paper is an extension of neutrosophic semigroup to soft semigroup. We also extend neutrosophic bisemigroup, neutrosophic  $N$ -semigroup to soft neutrosophic bisemigroup, and soft neutrosophic  $N$ -semigroup. Their related properties and results are explained with many illustrative examples, the notions related with strong part of neutrosophy also established within soft semigroup.

## References

- [1] H. Aktas and N. Cagman, Soft sets and soft groups, Inf. Sci., **177**(2007), 2726-2735.
- [2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets syst, **64**(1986), No. 2, 87-96.
- [3] M. Ali, F. Smarandache, M. Shabir and M. Naz, Soft neutrosophic bigroup and soft neutrosophic  $N$ -group, Neutrosophic Sets and Systems (Accepted).

- [4] M. I. Ali, F. Feng, X. Liu, W. K. Min and M. Shabir, On some new operations in soft set theory, *Comp. Math. Appl.*, **57**(2009), 1547-1553.
- [5] S. Broumi and F. Smarandache, Intuitionistic neutrosophic soft set, *J. Inf. & Comput. Sc.*, **8**(2013), 130-140.
- [6] D. Chen, E. C. C. Tsang, D. S. Yeung and X. Wang, The parameterization reduction of soft sets and its applications, *Comput. Math. Appl.*, **49**(2005), 757-763.
- [7] F. Feng, M. I. Ali and M. Shabir, Soft relations applied to semigroups, *Filomat*, **27**(2013), No. 7, 1183-1196.
- [8] M. B. Gorzalcany, A method of inference in approximate reasoning based on interval-valued fuzzy sets, *Fuzzy Sets Syst.*, **21**(1987), 1-17.
- [9] W. B. V. Kandasamy and F. Smarandache, Basic neutrosophic algebraic structures and their applications to fuzzy and neutrosophic models, *Hexis*, 2004.
- [10] W. B. V. Kandasamy and F. Smarandache, *N*-algebraic structures and *S-N*-algebraic structures, *Hexis Phoenix*, 2006.
- [11] W. B. V. Kandasamy and F. Smarandache, Some neutrosophic algebraic structures and neutrosophic *N*-algebraic structures, *Hexis*, 2006.
- [12] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, *Comput. Math. Appl.*, **45**(2003), 555-562.
- [13] P. K. Maji, Neutrosophic soft sets, *Ann. Fuzzy Math. Inf.*, **5**(2013), No. 1, 2093-9310.
- [14] D. Molodtsov, Soft set theory first results, *Comput. Math. Appl.*, **37**(1999), 19-31.
- [15] Z. Pawlak, Rough sets, *Int. J. Inf. Comp. Sci.*, **11**(1982), 341-356.
- [16] F. Smarandache, A unifying field in logics, *Neutrosophy: Neutrosophic probability, set and logic*, Rehoboth: American Research Press, 1999.
- [17] M. Shabir, M. Ali, M. Naz and F. Smarandache, Soft neutrosophic group, *Neutrosophic Sets and Systems.*, **1**(2013), 5-11.
- [18] L. A. Zadeh, Fuzzy sets, *Inf. Cont.*, **8**(1965), 338-353.

# Some remarks on fuzzy $\sigma$ -Baire spaces

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**Abstract** In this paper we investigate several characterizations of fuzzy  $\sigma$ -Baire spaces and study the conditions under which a fuzzy topological space becomes a fuzzy  $\sigma$ -Baire space.

**Keywords** Fuzzy dense set, fuzzy  $\sigma$ -nowhere dense set, fuzzy Baire space, fuzzy  $\sigma$ -Baire space, fuzzy almost resolvable space, fuzzy submaximal space, fuzzy Volterra space, fuzzy weakly Volterra space.

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## §1. Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced by L. A. Zadeh<sup>[17]</sup> in 1965. This concept provides a natural foundation for treating mathematically the fuzzy phenomena, which exist pervasively in our real world. The first notion of fuzzy topological spaces had been defined by C. L. Chang<sup>[3]</sup> in 1968 and this paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed.

The concepts of Baire spaces have been studied extensively in classical topology<sup>[4],[5],[7],[18]</sup>. The concept of Baire spaces in fuzzy setting was introduced and studied by G. Thangaraj and S. Anjalmose in [9]. The concept of  $\sigma$ -nowhere dense set was introduced and studied by Jiling Cao and Sina Greenwood<sup>[6]</sup>. The concept of fuzzy  $\sigma$ -Baire space was introduced and studied by G. Thangaraj and E. Poongothai in [11]. In this paper, we investigate several characterizations of fuzzy  $\sigma$ -Baire spaces and study the inter-relations between fuzzy  $\sigma$ -Baire spaces, fuzzy Baire spaces, fuzzy almost resolvable spaces, fuzzy Volterra spaces and fuzzy weakly Volterra spaces.

## §2. Preliminaries

By a fuzzy topological space we shall mean a non - empty set  $X$  together with a fuzzy topology  $T$  ( in the sense of CHANG ) and denote it by  $(X, T)$ .

**Definition 2.1.** Let  $\lambda$  and  $\mu$  be any two fuzzy sets in  $(X, T)$ . Then we define  $\lambda \vee \mu : X \rightarrow [0, 1]$  as follows :  $(\lambda \vee \mu)(x) = \text{Max}\{\lambda(x), \mu(x)\}$ . Also we define  $\lambda \wedge \mu : X \rightarrow [0, 1]$  as follows :  $(\lambda \wedge \mu)(x) = \text{Min}\{\lambda(x), \mu(x)\}$ .

For a family  $\{\lambda_i/i \in I\}$  of fuzzy sets in  $(X, T)$ , the union  $\psi = \vee_i \lambda_i$  and intersection  $\delta = \wedge_i \lambda_i$  are defined respectively as  $\psi(x) = \sup_i \{\lambda_i(x), x \in X\}$  and  $\delta(x) = \inf_i \{\lambda_i(x), x \in X\}$ .

**Definition 2.2.** Let  $(X, T)$  be a fuzzy topological space. For a fuzzy set  $\lambda$  of  $X$ , the interior  $\text{int}(\lambda)$  and the closure  $\text{cl}(\lambda)$  of  $X$  are defined respectively as

- (i)  $\text{int}(\lambda) = \vee \{\mu/\mu \leq \lambda, \mu \in T\}$ ,
- (ii)  $\text{cl}(\lambda) = \wedge \{\mu/\lambda \leq \mu, 1 - \mu \in T\}$ .

**Lemma 2.1.**<sup>[1]</sup> For any fuzzy set  $\lambda$  in a fuzzy topological space  $(X, T)$ ,

- (i)  $1 - \text{cl}(\lambda) = \text{int}(1 - \lambda)$ ,
- (ii)  $1 - \text{int}(\lambda) = \text{cl}(1 - \lambda)$ .

**Definition 2.3.**<sup>[8]</sup> A fuzzy set  $\lambda$  in a fuzzy topological space  $(X, T)$  is called fuzzy dense if there exists no fuzzy closed set  $\mu$  in  $(X, T)$  such that  $\lambda < \mu < 1$ .

**Definition 2.4.**<sup>[8]</sup> A fuzzy set  $\lambda$  in a fuzzy topological space  $(X, T)$  is called fuzzy nowhere dense if there exists no non-zero fuzzy open set  $\mu$  in  $(X, T)$  such that  $\mu < \text{cl}(\lambda)$ . That is,  $\text{intcl}(\lambda) = 0$ .

**Definition 2.5.**<sup>[2]</sup> A fuzzy set  $\lambda$  in a fuzzy topological space  $(X, T)$  is called a fuzzy  $F_\sigma$ -set in  $(X, T)$  if  $\lambda = \vee_{i=1}^\infty (\lambda_i)$  where  $1 - \lambda_i \in T$  for  $i \in I$ .

**Definition 2.6.**<sup>[2]</sup> A fuzzy set  $\lambda$  in a fuzzy topological space  $(X, T)$  is called a fuzzy  $G_\delta$ -set in  $(X, T)$  if  $\lambda = \wedge_{i=1}^\infty (\lambda_i)$  where  $\lambda_i \in T$  for  $i \in I$ .

**Definition 2.7.**<sup>[11]</sup> A fuzzy set  $\lambda$  in a fuzzy topological space  $(X, T)$  is called fuzzy  $\sigma$ -nowhere dense if  $\lambda$  is a fuzzy  $F_\sigma$ -set in  $(X, T)$  such that  $\text{int}(\lambda) = 0$ .

**Definition 2.8.**<sup>[9]</sup> A fuzzy topological space  $(X, T)$  is called a fuzzy Baire space if  $\text{int}(\vee_{i=1}^\infty (\lambda_i)) = 0$  where  $\lambda_i$ 's are fuzzy nowhere dense sets in  $(X, T)$ .

**Definition 2.9.**<sup>[8]</sup> A fuzzy set  $\lambda$  in a fuzzy topological space  $(X, T)$  is called a fuzzy first category set if  $\lambda = \vee_{i=1}^\infty (\lambda_i)$  where  $\lambda_i$ 's are fuzzy nowhere dense sets in  $(X, T)$ . Any other fuzzy set in  $(X, T)$  is said to be of fuzzy second category.

**Definition 2.10.**<sup>[11]</sup> A fuzzy set  $\lambda$  in a fuzzy topological space  $(X, T)$  is called a fuzzy  $\sigma$ -first category set if  $\lambda = \vee_{i=1}^\infty (\lambda_i)$  where  $\lambda_i$ 's are fuzzy  $\sigma$ -nowhere dense sets in  $(X, T)$ . Any other fuzzy set in  $(X, T)$  is said to be of fuzzy  $\sigma$ -second category.

**Definition 2.11.**<sup>[11]</sup> Let  $\lambda$  be a fuzzy  $\sigma$ -first category set in a fuzzy topological space  $(X, T)$ . Then  $1 - \lambda$  is called a fuzzy  $\sigma$ -residual set in  $(X, T)$ .

**Lemma 2.2.**<sup>[1]</sup> For a family of  $\{\lambda_\alpha\}$  of fuzzy sets of a fuzzy topological space  $X$ ,  $\vee \text{cl}(\lambda_\alpha) \leq \text{cl}(\vee \lambda_\alpha)$ . In case is a finite set,  $\vee \text{cl}(\lambda_\alpha) = \text{cl}(\vee \lambda_\alpha)$ . Also  $\vee \text{int}(\lambda_\alpha) \leq \text{int}(\vee \lambda_\alpha)$ .

### §3. Fuzzy $\sigma$ -Baire spaces

**Definition 3.1.**<sup>[11]</sup> Let  $(X, T)$  be a fuzzy topological space. Then  $(X, T)$  is called a fuzzy  $\sigma$ -Baire space if  $\text{int}(\vee_{i=1}^\infty (\lambda_i)) = 0$ , where  $\lambda_i$ 's are fuzzy  $\sigma$ -nowhere dense sets in  $(X, T)$ .

**Theorem 3.1.**<sup>[11]</sup> Let  $(X, T)$  be a fuzzy topological space. Then the following are equivalent:



- (1)  $(X, T)$  is a fuzzy  $\sigma$ -Baire space.
- (2)  $\text{int}(\lambda) = 0$  for every fuzzy  $\sigma$ -first category set  $\lambda$  in  $(X, T)$ .
- (3)  $\text{cl}(\mu) = 1$  for every fuzzy  $\sigma$ -residual set  $\mu$  in  $(X, T)$ .

**Definition 3.2.**<sup>[16]</sup> A fuzzy topological space  $(X, T)$  is called a fuzzy almost resolvable space if  $\bigvee_{i=1}^{\infty}(\lambda_i) = 1$ , where the fuzzy sets  $\lambda_i$ 's in  $(X, T)$  are such that  $\text{int}(\lambda_i) = 0$ . Otherwise  $(X, T)$  is called a fuzzy almost irresolvable space.

**Definition 3.3.** A fuzzy topological space  $(X, T)$  is called a fuzzy  $\sigma$ -first category space if the fuzzy set  $1_X$  is a fuzzy  $\sigma$ -first category set in  $(X, T)$ . That is,  $1_X = \bigvee_{i=1}^{\infty}(\lambda_i)$ , where  $\lambda_i$ 's are fuzzy  $\sigma$ -nowhere dense sets in  $(X, T)$ . Otherwise  $(X, T)$  will be called a fuzzy  $\sigma$ -second category space in  $(X, T)$ .

**Proposition 3.1.** If the fuzzy topological space  $(X, T)$  is a fuzzy  $\sigma$ -first category space, then  $(X, T)$  is a fuzzy almost resolvable space.

**Proof.** Let the fuzzy topological space  $(X, T)$  be a fuzzy  $\sigma$ -first category space. Then we have  $\bigvee_{i=1}^{\infty}(\lambda_i) = 1$ , where  $\lambda_i$ 's are fuzzy  $\sigma$ -nowhere dense sets in  $(X, T)$ . Since  $\lambda_i$  is a fuzzy  $\sigma$ -nowhere dense set in  $(X, T)$ ,  $\lambda_i$  is a fuzzy  $F_{\sigma}$ -set in  $(X, T)$  and  $\text{int}(\lambda_i) = 0$ . Hence  $\bigvee_{i=1}^{\infty}(\lambda_i) = 1$ , where  $\text{int}(\lambda_i) = 0$  and therefore  $(X, T)$  is a fuzzy almost resolvable space.

**Theorem 3.2.**<sup>[11]</sup> If the fuzzy topological space  $(X, T)$  is a fuzzy  $\sigma$ -Baire space, then  $(X, T)$  is a fuzzy  $\sigma$ -second category space.

**Proposition 3.2.** If the fuzzy topological space  $(X, T)$  is a fuzzy  $\sigma$ -Baire space, then  $(X, T)$  is a fuzzy almost irresolvable space.

**Proof.** Let  $(X, T)$  be a fuzzy  $\sigma$ -Baire space. Since every fuzzy  $\sigma$ -Baire space is a fuzzy  $\sigma$ -second category space,  $(X, T)$  is not a fuzzy  $\sigma$ -first category space. Then  $\bigvee_{i=1}^{\infty}(\lambda_i) \neq 1$ , where  $\lambda_i$ 's are fuzzy  $\sigma$ -nowhere dense sets in  $(X, T)$ . Now  $\lambda_i$ 's are fuzzy  $\sigma$ -nowhere dense sets in  $(X, T)$ , implies that  $\text{int}(\lambda_i) = 0$ , ( $i = 1$  to  $\infty$ ). Hence  $\bigvee_{i=1}^{\infty}(\lambda_i) \neq 1$ , where  $\text{int}(\lambda_i) = 0$  and therefore  $(X, T)$  is a fuzzy almost irresolvable space.

**Definition 3.4.**<sup>[13]</sup> A fuzzy topological space  $(X, T)$  is called a fuzzy strongly irresolvable space if  $\text{clint}(\lambda) = 1$  for each fuzzy dense set  $\lambda$  in  $(X, T)$ .

**Proposition 3.3.** If the fuzzy topological space  $(X, T)$  is a fuzzy strongly irresolvable, fuzzy  $\sigma$ -Baire space and  $\lambda$  is a fuzzy  $\sigma$ -first category set in  $(X, T)$ , then  $\lambda$  is a fuzzy nowhere dense set in  $(X, T)$ .

**Proof.** Let  $\lambda$  be a fuzzy  $\sigma$ -first category set in  $(X, T)$ . Then  $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$ , where  $\lambda_i$ 's are fuzzy  $\sigma$ -nowhere dense sets in  $(X, T)$ . Since  $(X, T)$  is a fuzzy  $\sigma$ -Baire space, by theorem 3.1,  $\text{int}(\lambda) = 0$  in  $(X, T)$ . Then we have  $1 - \text{int}(\lambda) = 1$ . This implies that  $\text{cl}(1 - \lambda) = 1$ . Since  $(X, T)$  is a fuzzy strongly irresolvable space, for the fuzzy dense set  $1 - \lambda$  in  $(X, T)$ , we have  $\text{clint}(1 - \lambda) = 1$ . Then  $1 - \text{intcl}(\lambda) = 1$  and hence  $\text{intcl}(\lambda) = 0$ . Therefore  $\lambda$  is a fuzzy nowhere dense set in  $(X, T)$ .

**Proposition 3.4.** If  $\lambda_i$ 's are the fuzzy  $\sigma$ -first category sets in a fuzzy strongly irresolvable, fuzzy  $\sigma$ -Baire space  $(X, T)$ , then  $\bigvee_{i=1}^{\infty}(\lambda_i)$  is a fuzzy first category set in  $(X, T)$ .

**Proof.** Let  $\lambda_i$ 's be the fuzzy  $\sigma$ -first category sets in a fuzzy strongly irresolvable, fuzzy  $\sigma$ -Baire space  $(X, T)$ . Then, by Proposition 3.3,  $\lambda_i$ 's are fuzzy nowhere dense sets in  $(X, T)$  and hence  $\bigvee_{i=1}^{\infty}(\lambda_i)$  is a fuzzy first category set in  $(X, T)$ .

**Proposition 3.5.** If  $\text{int}(\bigvee_{i=1}^{\infty} (\lambda_i)) = 0$ , where  $\lambda_i$ 's are the fuzzy  $\sigma$ -first category sets in a fuzzy strongly irresolvable, fuzzy  $\sigma$ -Baire space  $(X, T)$ , then  $(X, T)$  is a fuzzy Baire space.

**Proof.** Let  $\lambda_i$ 's be the fuzzy  $\sigma$ -first category sets in a fuzzy strongly irresolvable, fuzzy  $\sigma$ -Baire space  $(X, T)$ . Then, by Proposition 3.4,  $\bigvee_{i=1}^{\infty} (\lambda_i)$  is a fuzzy first category set in  $(X, T)$ . Then  $\bigvee_{i=1}^{\infty} (\lambda_i) = \bigvee_{j=1}^{\infty} (\mu_j)$ , where the fuzzy sets  $\mu_j$ 's are fuzzy nowhere dense sets in  $(X, T)$ . Now  $\text{int}(\bigvee_{i=1}^{\infty} (\lambda_i)) = 0$ , implies that  $\text{int}(\bigvee_{j=1}^{\infty} (\mu_j)) = 0$ , where the fuzzy sets  $\mu_j$ 's are fuzzy nowhere dense sets in  $(X, T)$ . Hence the fuzzy topological space  $(X, T)$  is a fuzzy Baire space.

**Definition 3.5.**<sup>[2]</sup> A fuzzy topological space  $(X, T)$  is called a fuzzy submaximal space if for each fuzzy set  $\lambda$  in  $(X, T)$  such that  $cl(\lambda) = 1$ , then  $\lambda \in T$  in  $(X, T)$ .

**Theorem 3.3.**<sup>[13]</sup> If the fuzzy topological space  $(X, T)$  is a fuzzy submaximal space, then  $(X, T)$  is a fuzzy strongly irresolvable space.

**Proposition 3.6.** If the fuzzy topological space  $(X, T)$  is a fuzzy submaximal, fuzzy  $\sigma$ -Baire space and  $\lambda$  is a fuzzy  $\sigma$ -first category set in  $(X, T)$ , then  $\lambda$  is a fuzzy nowhere dense set in  $(X, T)$ .

**Proof.** Let  $\lambda$  be a fuzzy  $\sigma$ -first category set in a fuzzy submaximal, fuzzy  $\sigma$ -Baire space. Since  $(X, T)$  is a fuzzy submaximal space, by theorem 3.3,  $(X, T)$  is a fuzzy strongly irresolvable space. Then  $(X, T)$  is a fuzzy strongly irresolvable, fuzzy  $\sigma$ -Baire space. Since  $\lambda$  is a fuzzy  $\sigma$ -first category set in  $(X, T)$ , by proposition 3.3,  $\lambda$  is a fuzzy nowhere dense set in  $(X, T)$ .

**Proposition 3.7.** If  $\lambda_i$ 's are the fuzzy  $\sigma$ -first category sets in a fuzzy submaximal, fuzzy  $\sigma$ -Baire space  $(X, T)$ , then  $\bigvee_{i=1}^{\infty} (\lambda_i)$  is a fuzzy first category set in  $(X, T)$ .

**Proof.** Let  $\lambda_i$ 's be the fuzzy  $\sigma$ -first category sets in a fuzzy submaximal, fuzzy  $\sigma$ -Baire space  $(X, T)$ . Then, by proposition 3.6,  $\lambda_i$ 's are fuzzy nowhere dense sets in  $(X, T)$  and hence  $\bigvee_{i=1}^{\infty} (\lambda_i)$  is a fuzzy first category set in  $(X, T)$ .

**Proposition 3.8.** If  $\text{int}(\bigvee_{i=1}^{\infty} (\lambda_i)) = 0$ , where  $\lambda_i$ 's are the fuzzy  $\sigma$ -first category sets in a fuzzy submaximal, fuzzy  $\sigma$ -Baire space  $(X, T)$ , then  $(X, T)$  is a fuzzy Baire space.

**Proof.** Let  $\lambda_i$ 's be the fuzzy  $\sigma$ -first category sets in a fuzzy submaximal, fuzzy  $\sigma$ -Baire space  $(X, T)$ . Then, by proposition 3.7,  $\bigvee_{i=1}^{\infty} (\lambda_i)$  is a fuzzy first category set in  $(X, T)$ . Then  $\bigvee_{i=1}^{\infty} (\lambda_i) = \bigvee_{j=1}^{\infty} (\mu_j)$ , where the fuzzy sets  $\mu_j$ 's are fuzzy nowhere dense sets in  $(X, T)$ . Now  $\text{int}(\bigvee_{i=1}^{\infty} (\lambda_i)) = 0$ , implies that  $\text{int}(\bigvee_{j=1}^{\infty} (\mu_j)) = 0$ , where the fuzzy sets  $\mu_j$ 's are fuzzy nowhere dense sets in  $(X, T)$ . Hence  $(X, T)$  is a fuzzy Baire space.

**Definition 3.6.**<sup>[14]</sup> A fuzzy topological space  $(X, T)$  is called a fuzzy Volterra space if  $cl(\bigwedge_{i=1}^N (\lambda_i)) = 1$ , where  $\lambda_i$ 's are fuzzy dense and fuzzy  $G_\delta$ -sets in  $(X, T)$ .

**Definition 3.7.**<sup>[14]</sup> Let  $(X, T)$  be a fuzzy topological space. Then  $(X, T)$  is called a fuzzy weakly Volterra space if  $cl(\bigwedge_{i=1}^N (\lambda_i)) \neq 0$ , where  $\lambda_i$ 's are fuzzy dense and fuzzy  $G_\delta$ -sets in  $(X, T)$ .

**Proposition 3.9.** If  $\lambda_i$ 's ( $i = 1$  to  $N$ ) are fuzzy  $\sigma$ -nowhere dense sets in a fuzzy topological space  $(X, T)$  and  $\text{int}(\bigvee_{i=1}^N (\lambda_i)) = 0$ , then  $(X, T)$  is a fuzzy Volterra space.

**Proof.** Let  $\lambda_i$ 's ( $i = 1$  to  $N$ ) be fuzzy  $\sigma$ -nowhere dense sets in  $(X, T)$ . Then  $\lambda_i$ 's are fuzzy  $F_\sigma$ -sets with  $\text{int}(\lambda_i) = 0$  ( $i = 1$  to  $N$ ). Now  $1 - \text{int}(\lambda_i) = 1$ . Then, we have  $cl(1 - \lambda_i) = 1$ . That is,  $(1 - \lambda_i)$ 's are fuzzy dense sets in  $(X, T)$ . Since  $\lambda_i$ 's are fuzzy  $F_\sigma$ -sets,  $(1 - \lambda_i)$ 's are fuzzy  $G_\delta$ -sets in  $(X, T)$ . Hence  $(1 - \lambda_i)$ 's are fuzzy dense and fuzzy  $G_\delta$ -sets in  $(X, T)$ . Now,

$cl(\wedge_{i=1}^N (1 - \lambda_i)) = cl(1 - [\vee_{i=1}^N (\lambda_i)]) = 1 - int(\vee_{i=1}^N (\lambda_i)) = 1 - 0 = 1$ . Hence  $(X, T)$  is a fuzzy Volterra space.

**Theorem 3.4.**<sup>[12]</sup> In a fuzzy topological space  $(X, T)$ , a fuzzy set  $\lambda$  is a fuzzy  $\sigma$ -nowhere dense set in  $(X, T)$  if and only if  $1 - \lambda$  is a fuzzy dense and fuzzy  $G_\delta$ -set in  $(X, T)$ .

**Theorem 3.5.**<sup>[15]</sup> If the fuzzy topological space  $(X, T)$  is a fuzzy almost irresolvable space, then  $(X, T)$  is a fuzzy weakly Volterra space.

**Theorem 3.6.**<sup>[15]</sup> If the fuzzy topological space  $(X, T)$  is a fuzzy  $\sigma$ -second category space, then  $(X, T)$  is a fuzzy weakly Volterra space.

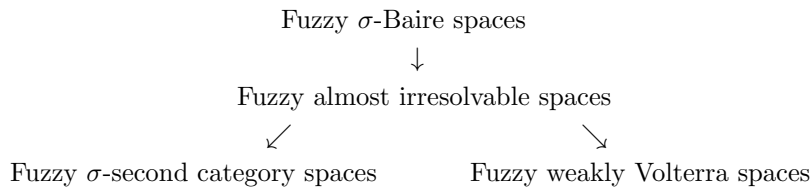
**Proposition 3.10.** If a fuzzy topological space  $(X, T)$  is not a fuzzy weakly Volterra space, then  $(X, T)$  is a fuzzy  $\sigma$ -first category space.

**Proof.** Let  $(X, T)$  be a fuzzy non-weakly Volterra space. Then, we have  $cl(\wedge_{i=1}^N (\lambda_i)) = 0$ , where  $\lambda_i$ 's are fuzzy dense and fuzzy  $G_\delta$ -sets in  $(X, T)$ . Since  $\lambda_i$ 's are fuzzy dense and fuzzy  $G_\delta$ -sets in  $(X, T)$ , by theorem 3.4,  $(1 - \lambda_i)$ 's are fuzzy  $\sigma$ -nowhere dense sets in  $(X, T)$ . Let  $\mu_j$ 's ( $j = 1$  to  $\infty$ ) be fuzzy  $\sigma$ -nowhere dense sets in  $(X, T)$  in which the first  $N$  fuzzy  $\sigma$ -nowhere dense sets be  $(1 - \lambda_i)$ 's. Now  $\vee_{i=1}^N (1 - \lambda_i) \leq \vee_{j=1}^\infty (\mu_j)$ . Then  $1 - \wedge_{i=1}^N (\lambda_i) \leq \vee_{j=1}^\infty (\mu_j)$  and hence  $int[1 - \wedge_{i=1}^N (\lambda_i)] \leq int[\vee_{j=1}^\infty (\mu_j)]$ . This implies that  $1 - cl[\wedge_{i=1}^N (\lambda_i)] \leq int[\vee_{j=1}^\infty (\mu_j)]$ . Then  $1 - int[\vee_{j=1}^\infty (\mu_j)] \leq cl[\wedge_{i=1}^N (\lambda_i)]$ . Now  $1 - [\vee_{j=1}^\infty (\mu_j)] \leq 1 - int[\vee_{j=1}^\infty (\mu_j)] \leq cl[\wedge_{i=1}^N (\lambda_i)]$ . Since  $cl[\wedge_{i=1}^N (\lambda_i)] = 0$ ,  $1 - [\vee_{j=1}^\infty (\mu_j)] = 0$ . Then,  $[\vee_{j=1}^\infty (\mu_j)] = 1$ , where  $\mu_j$ 's ( $j = 1$  to  $\infty$ ) are fuzzy  $\sigma$ -nowhere dense sets in  $(X, T)$ . Hence  $(X, T)$  is a fuzzy  $\sigma$ -first category space.

**Proposition 3.11.** If a fuzzy topological space  $(X, T)$  is a fuzzy weakly Volterra space, then  $(X, T)$  is not a fuzzy  $\sigma$ -Baire space.

**Proof.** Let  $(X, T)$  be a fuzzy weakly Volterra space. Then, we have  $cl(\wedge_{i=1}^N (\lambda_i)) \neq 0$  where  $\lambda_i$ 's are fuzzy dense and fuzzy  $G_\delta$ -sets in  $(X, T)$ . Since  $\lambda_i$ 's are fuzzy dense and fuzzy  $G_\delta$ -sets in  $(X, T)$ , by theorem 3.4,  $(1 - \lambda_i)$ 's are fuzzy  $\sigma$ -nowhere dense sets in  $(X, T)$ . Let  $\mu_j$ 's ( $j = 1$  to  $\infty$ ) be fuzzy  $\sigma$ -nowhere dense sets in  $(X, T)$  in which the first  $N$  fuzzy  $\sigma$ -nowhere dense sets be  $(1 - \lambda_i)$ 's. Now  $\vee_{i=1}^N (1 - \lambda_i) \leq \vee_{j=1}^\infty (\mu_j)$ . Then  $1 - \wedge_{i=1}^N (\lambda_i) \leq \vee_{j=1}^\infty (\mu_j)$  and hence  $int[1 - \wedge_{i=1}^N (\lambda_i)] \leq int[\vee_{j=1}^\infty (\mu_j)]$ . This implies that  $1 - cl[\wedge_{i=1}^N (\lambda_i)] \leq int[\vee_{j=1}^\infty (\mu_j)]$ . Since  $cl[\wedge_{i=1}^N (\lambda_i)] \neq 0$ ,  $int[\vee_{j=1}^\infty (\mu_j)] \neq 0$  where  $\mu_j$ 's ( $j = 1$  to  $\infty$ ) are fuzzy  $\sigma$ -nowhere dense sets in  $(X, T)$ . Hence  $(X, T)$  is not a fuzzy  $\sigma$ -Baire space.

**Remarks.** The inter-relation between fuzzy  $\sigma$ -Baire spaces, fuzzy almost irresolvable spaces, fuzzy weakly Volterra spaces and fuzzy  $\sigma$ -second category spaces can be summarized as follows :



But, Fuzzy  $\sigma$ -second category spaces  $\nrightarrow$  Fuzzy  $\sigma$ -Baire spaces

and

Fuzzy weakly Volterra spaces  $\nrightarrow$  Fuzzy  $\sigma$ -Baire spaces.

## References

- [1] K. K. Azad, On fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity, *J. Math. Anal. Appl.*, **82**(1981), 14-32.
- [2] G. Balasubramanian, Maximal fuzzy topologies, *Kybernetika*, **31**(1995), No. 5, 459-464.
- [3] C. L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.*, **24**(1968), 182-190.
- [4] G. Gruenhage and D. Lutzer, Baire and Volterra spaces, *Proc. Amer. Soc.*, **128**(2000), 311-312.
- [5] R. C. Haworth and R. A. McCoy, Baire spaces, *Dissertationes Math.*, **141**(1977), 77.
- [6] Jiling Cao and Sina Greenwood, The ideal generated by  $\sigma$ -nowhere dense sets, *Applied General Topology*, **1**(2000), 1-3.
- [7] T. Neubrunn, A note on mappings of Baire spaces, *Math. Slovaca*, **27**(1977), No. 2, 173-176.
- [8] G. Thangaraj and G. Balasubramanian, On somewhat fuzzy continuous functions, *J. Fuzzy Math.*, **11**(2003), No. 2, 725-736.
- [9] G. Thangaraj and S. Anjalmose, On fuzzy Baire spaces, *J. Fuzzy Math.*, **21**(2013), No. 3, 667-676.
- [10] G. Thangaraj and S. Anjalmose, A note on fuzzy Baire spaces, *Inter. J. Fuzzy Math. and Sys.*, **3**(2013), No. 4, 269-274.
- [11] G. Thangaraj and E. Poongothai, On fuzzy  $\sigma$ -Baire spaces, *Inter. J. Fuzzy Math. and Sys.*, **3**(2013), No. 4, 275-283.
- [12] G. Thangaraj and E. Poongothai, A note on fuzzy  $\sigma$ -Baire spaces ( submitted to *Ann. Fuzzy Math. Inform* ).
- [13] G. Thangaraj and V. Seenivasan, On fuzzy strongly irresolvable spaces, *Proc. Nat. Conf. on Fuzzy Math. and Graph Theory*, Jamal Mohamed College, Tiruchi, Tamil Nadu, India, March 27-28, (2008), 1-6.
- [14] G. Thangaraj and S. Soundararajan, On fuzzy Volterra spaces, *J. Fuzzy Math.*, **21**(2013), No. 4, 895-904.
- [15] G. Thangaraj and S. Soundararajan, A note on fuzzy weakly Volterra spaces, *Inter. J. Statistika and Matematika*. **9**(2014), No. 1, 20-24.
- [16] G. Thangaraj and D. Vijayan, On fuzzy almost resolvable and fuzzy almost irresolvable spaces, *Inter. J. Statistika and Matematika*, **9**(2014), No. 2, 61-65.
- [17] L. A. Zadeh, Fuzzy sets, *Information and Control*, **8** (1965), 338-353.
- [18] Zdenek Frolik, Baire spaces and some generalizations of complete metric spaces, *Czech. Math. Journal*, **11**(1961), No. 2, 237-248.

The background is a deep red with a subtle texture. On the left, there are intricate, swirling red lines that resemble calligraphic flourishes or smoke. On the right, there is a large, stylized, light-red figure that appears to be a person in a dynamic pose, possibly a dancer or a warrior, with flowing lines suggesting movement. A bright, vertical light beam or lens flare effect runs down the right side of the image, adding a sense of depth and focus.

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